

Characteristic cycles and the conductor of direct image

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Abstract

We prove the functoriality for proper push-forward of the characteristic cycles of constructible complexes by morphisms of smooth projective schemes over a perfect field, under the assumption that the direct image of the singular support has the dimension at most that of the target of the morphism. The functoriality is deduced from a conductor formula which is a special case for morphisms to curves. The conductor formula in the constant coefficient case gives the geometric case of a formula conjectured by Bloch.

Let k be a perfect field and Λ be a finite field of characteristic invertible in k . For a constructible complex \mathcal{F} of Λ -modules on a smooth scheme X over k , the characteristic cycle $CC\mathcal{F}$ is defined in [16, Definition 5.10] as a cycle supported on the singular support $SS\mathcal{F}$ defined by Beilinson in [2] as a closed conical subset of the cotangent bundle T^*X . We study the functoriality of characteristic cycles for proper push-forward.

Let $f: X \rightarrow Y$ be a morphism of smooth projective schemes over k . Then, we prove in Theorem 2.2.5 the equality

$$(2.11) \quad CCRf_*\mathcal{F} = f_!CC\mathcal{F}$$

conjectured in [17, Conjecture 1] under the assumption $\dim f_*SS\mathcal{F} \leq \dim Y$ for the direct image $f_*SS\mathcal{F} \subset T^*Y$. The precise definitions will be given in Subsection 2.1. We can slightly weaken the assumption, as is seen in Theorem 2.2.5. The formula (2.11) is an algebraic analogue of [12, Proposition 9.4.2] where functorial properties of characteristic cycles are studied in a transcendental context. In the case where $Y = \text{Spec } k$, the equality (2.11) is the index formula

$$(2.12) \quad \chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T_X^*X)_{T^*X}$$

computing the Euler-Poincaré characteristic as an intersection number proved in [16, Theorem 7.13].

We deduce the functoriality (2.11) from the index formula (2.12) in Subsection 2.2 as follows. By taking a projective embedding of Y and a good pencil, we reduce it to the case where Y is a projective smooth curve. By the index formula (2.12) applied to a general fiber, the equality (2.11) is equivalent to a conductor formula

$$(2.17) \quad -a_y Rf_*\mathcal{F} = (CC\mathcal{F}, df)_{T^*X, X_y}$$

proved in Theorem 2.2.3, where the left hand side denotes the Artin conductor at a closed point $y \in Y$ of the direct image. In the case where \mathcal{F} is the constant sheaf Λ , the right

hand side equals the localized self-intersection product defined in [4] and the formula (2.17) specializes to the geometric case, Corollary 2.2.4, of the conductor formula conjectured in [4] by Bloch.

Further the index formula implies that we have an equality (2.18) for the sums over $y \in Y$ of the both sides in (2.17). To deduce (2.17) from (2.18) for the sums, it suffices to show the existence of a covering of Y étale at a fixed point y killing the contributions of the other points.

For the vanishing of the left hand side, the local acyclicity of $f: X \rightarrow Y$ relative to \mathcal{F} is a sufficient condition. The $SS\mathcal{F}$ -transversality of $f: X \rightarrow Y$ defined in Definition 1.3.3 and studied in Subsection 1.3 after some preliminaries in Subsection 1.2 is a stronger condition and is a sufficient condition for the vanishing of the right hand side. Thus, the proof of (2.17) is reduced to showing variants of the stable reduction theorem on the existence of ramified covering of Y such that the base change of $f: X \rightarrow Y$ is locally acyclic relatively to a modification \mathcal{F}' of the pull-back and is $SS\mathcal{F}'$ -transversal.

We show that $f: X \rightarrow Y$ is locally acyclic relatively to a modification of a perverse sheaf \mathcal{F} if the inertia action on the nearby cycles complex $R\Psi\mathcal{F}$ is trivial in Proposition 1.1.2.2. This is rather a direct consequence of the relation of the direct image by the open immersion of the generic fiber with the nearby cycles complex. As we work with torsion coefficients, the condition is satisfied over a ramified covering of Y .

Further, we show that the local acyclicity of $f: X \rightarrow Y$ relatively to \mathcal{F} implies the existence of a ramified covering such that the base change of $f: X \rightarrow Y$ is $SS\mathcal{F}'$ -transversal for the pull-back \mathcal{F}' of \mathcal{F} in Corollary 1.5.4 of Theorem 1.5.2. Theorem 1.5.2 is deduced from a weaker version Proposition 1.4.4 which is proved by using the alteration [5, Theorem 8.2]. In Proposition 1.4.4, the ramified covering may inseparable, while it is generically étale in Theorem 1.5.2. Theorem 1.5.2 is proved by an argument similar to that in the proof of [6, Proposition 3.2] by using a consequence of the stable reduction theorem [18, Theorem 1.5].

We also prove an index formula Proposition 2.3.3 for vanishing cycles complex.

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1 Local acyclicity and transversality

1.1 Nearby cycles and local acyclicity

We fix some conventions on perverse sheaves. Let X be a noetherian scheme and let Λ be a finite field of characteristic ℓ invertible on X . We say that a complex \mathcal{F} of Λ -modules on the étale site of X is constructible if the cohomology sheaf $\mathcal{H}^q \mathcal{F}$ is constructible for every integer q and $\mathcal{H}^q \mathcal{F} = 0$ except for finitely many q . Let $D_c^b(X, \Lambda)$ denote the category of constructible complexes of Λ -modules.

First we recall the case where X is a scheme of finite type over a field k . Let Λ be a finite field of characteristic ℓ invertible in k . Then, the t -structure $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ on $D_c^b(X, \Lambda)$ relative to the middle perversity is defined in [3, 2.2.10] and the perverse sheaves on X form an abelian subcategory $\text{Perv}(X, \Lambda) = {}^pD^{\leq 0} \cap {}^pD^{\geq 0}$.

Next, we consider the case where X is a scheme of finite type over the spectrum S of a discrete valuation ring as in [11, 4.6]. Let s and η denote the closed point and the generic point of S respectively and let $i: X_s \rightarrow X$ and $j: X_\eta \rightarrow X$ be the closed immersion and the open immersion of the fibers. Let Λ be a finite field of characteristic ℓ invertible on S . Then, we consider the t -structure on $D_c^b(X, \Lambda)$ obtained by gluing ([3, 1.4.10]) the t -structure $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ on $D_c^b(X_s, \Lambda)$ and the t -structure $({}^pD^{\leq -1}, {}^pD^{\geq -1})$ on $D_c^b(X_\eta, \Lambda)$. In particular, a constructible complex $\mathcal{F} \in D_c^b(X, \Lambda)$ is contained in ${}^pD^{\leq 0}$ if and only if we have $i^* \mathcal{F} \in {}^pD^{\leq 0}$ and $j^* \mathcal{F} \in {}^pD^{\leq -1}$.

Note that if the t -structure on $D_c^b(X_\eta, \Lambda)$ where X_η is regarded as a scheme over η is $({}^pD^{\leq 0}, {}^pD^{\geq 0})$, then that on $D_c^b(X_\eta, \Lambda)$ where X_η is regarded as a scheme over S is $({}^pD^{\leq -1}, {}^pD^{\geq -1})$. To distinguish them, we call the former the t -structure on X_η over η and the latter the t -structure on X_η over S . We use the same terminology for perverse sheaves on X_η .

The functors $j_!, Rj_*: D_c^b(X_\eta, \Lambda) \rightarrow D_c^b(X, \Lambda)$ are t -exact with respect to the t -structure on X_η over S . This follows from [1, Théorème 3.1] by the argument in [11, 4.6 (a)]. Let $\mathcal{F} \in \text{Perv}(X_\eta, \Lambda)$ be a perverse sheaf on X_η over S . Then the intermediate extension $j_{!*} \mathcal{F} \in \text{Perv}(X, \Lambda)$ is defined as the image

$$j_{!*} \mathcal{F} = \text{Im}(j_! \mathcal{F} \rightarrow Rj_* \mathcal{F}).$$

We have ${}^p\mathcal{H}^q i^* Rj_* \mathcal{F} = 0$ for $q \neq 0, -1$ and the morphism $j_{!*} \mathcal{F} \rightarrow Rj_* \mathcal{F}$ induces an isomorphism

$$(1.1) \quad i^* j_{!*} \mathcal{F} \rightarrow {}^p\mathcal{H}^{-1} i^* Rj_* \mathcal{F}.$$

This is deduced similarly as [3, (4.1.12.1)] from a consequence [3, (4.1.11.1)] of the t -exactness of the functors $j_!$ and Rj_* .

Assume that S is strictly local. Let $\bar{\eta}$ be a geometric point above η and let $\bar{j}: X_{\bar{\eta}} \rightarrow X_\eta$ denote the canonical morphisms. Then, the nearby cycles functor

$$R\Psi = i^* R(j\bar{j})_* \bar{j}^*: D_c^b(X_\eta, \Lambda) \rightarrow D_c^b(X_s, \Lambda)$$

is t -exact with respect to the t -structure on X_η over η [11, Corollaire 4.5].

Lemma 1.1.1. *Let $S = \operatorname{Spec} \mathcal{O}_K$ be the spectrum of a strictly local discrete valuation ring and let s and η denote the closed and the generic point of S respectively. Let X be a scheme of finite type over S , and let $i: X_s \rightarrow X$ and $j: X_\eta \rightarrow X$ denote the immersions. Let \mathcal{F} be a perverse sheaf of Λ -modules on X_η over η . Then the morphism $i^* Rj_* \mathcal{F} \rightarrow R\Psi \mathcal{F}$ induces an isomorphism*

$$(1.2) \quad {}^p\mathcal{H}^0 i^* Rj_* \mathcal{F} \rightarrow (R\Psi \mathcal{F})^I$$

to the inertia fixed part as a perverse sheaf on X_s .

Proof. Let $P \subset I$ denote the wild inertia subgroup. Then, since the functor taking the P -invariant parts is an exact functor, we have an isomorphism $i^* Rj_* \mathcal{F} \rightarrow R\Gamma(I/P, (R\Psi \mathcal{F})^P)$. Since the profinite group I/P is cyclic, the assertion follows. \square

We study the local acyclicity of a morphism to the spectrum of a discrete valuation ring with respect to a perverse sheaf.

Proposition 1.1.2. *Let $S = \operatorname{Spec} \mathcal{O}_K$ be the spectrum of a discrete valuation ring and let s and η denote the closed and the generic point of S respectively. Let X be a scheme of finite type over S , and let $i: X_s \rightarrow X$ and $j: X_\eta \rightarrow X$ denote the immersions.*

1. *Let \mathcal{G} be a perverse sheaf of Λ -modules on X . If $X \rightarrow S$ is locally acyclic relatively to \mathcal{G} , then \mathcal{G} has no non-zero subquotient supported on the closed fiber and is isomorphic to $j_{!*} j^* \mathcal{G}$.*

2. *For a perverse sheaf \mathcal{F} of Λ -modules on X_η over S , the following conditions are equivalent:*

(1) *The morphism $X \rightarrow S$ is locally acyclic relatively to $j_{!*} \mathcal{F}$.*

(2) *Let \bar{s} be a geometric point above the closed point $s \in S$ and let $\bar{i}: X_{\bar{s}} \rightarrow X$ denote the canonical morphism. Then, the canonical morphism*

$$(1.3) \quad \bar{i}^* j_{!*} \mathcal{F} \rightarrow R\Psi \mathcal{F}$$

is an isomorphism.

(3) *The inertia group I of K acts trivially on the nearby cycles complex $R\Psi \mathcal{F}$.*

(4) *The formation of $j_{!*} \mathcal{F}$ commutes with the pull-back by faithfully flat morphisms $S' \rightarrow S$ of the spectra of discrete valuation rings.*

Proof. 1. The local acyclicity is equivalent to the vanishing $R\Phi \mathcal{G} = 0$. Since the shifted vanishing cycles functor $R\Phi[-1]: D_c^b(X, \Lambda) \rightarrow D_c^b(X_s, \Lambda)$ is t -exact [11, Corollaire 4.6], it is reduced to the case where \mathcal{G} is a simple perverse sheaf. If \mathcal{G} is supported on the closed fiber, we have $R\Phi \mathcal{G}[-1] = \mathcal{G}$ and the assertion follows.

2. (1) \Leftrightarrow (2): The condition (2) is equivalent to that for every geometric point x of X_s , the canonical morphism $j_{!*} \mathcal{F}_x \rightarrow R\Gamma(X_{(x)} \times_{S_{(s)}} \bar{\eta}, \mathcal{F})$ is an isomorphism.

(2) \Leftrightarrow (3): By (1.1) and (1.2), the morphism (1.3) induces an isomorphism $\bar{i}^* j_{!*} \mathcal{F} \rightarrow (R\Psi \mathcal{F})^I$.

(2) \Rightarrow (4): Since the formation of nearby cycles complex $R\Psi \mathcal{F}$ commutes with base change [6, Proposition 3.7], the isomorphism (1.3) implies the condition (4).

(4) \Rightarrow (2): There exists a finite extension K' of K such that the inertia action $I' \subset I$ on $R\Psi \mathcal{F}$ is trivial, since Λ is a finite field. Let $j': X_{K'} \rightarrow X_{S'}$ denote the base change of the open immersion j by $S' = \operatorname{Spec} \mathcal{O}_{K'} \rightarrow S$, let $i': X_{\bar{s}} \rightarrow X_{S'}$ denote the canonical morphism and let \mathcal{F}' denote the pull-back of \mathcal{F} on $X_{K'}$. We factorize the morphism (1.3)

as the composition of $\bar{i}^* j_{1*} \mathcal{F} \rightarrow \bar{i}'^* j'_{1*} \mathcal{F}' \rightarrow R\Psi \mathcal{F}$. By (3) \Rightarrow (2) already proven, the second arrow is an isomorphism. The condition (4) implies that the first arrow is an isomorphism. Hence the composition (1.3) is an isomorphism. \square

Finally, we consider the case where X is a scheme of finite type over a regular noetherian connected scheme S of dimension 1. Let Λ be a finite field of characteristic ℓ invertible on S . Then the t -structure $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ on $D_c^b(X, \Lambda)$ is defined as the intersection of the inverse images of the t -structures $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ on $D_c^b(X \times_S S_s, \Lambda)$ for the base changes by the localizations $S_s \rightarrow S$ at closed points $s \in S$. If $Y = S$ is a smooth curve over a field k and if $f: X \rightarrow Y$ is a morphism of schemes of finite type over k , the t -structure on $D_c^b(X, \Lambda)$ defined above is the same as that defined by considering X as a scheme of finite type over k .

Corollary 1.1.3. *Let S be a regular noetherian scheme of dimension 1. Let X be a scheme of finite type over S and \mathcal{F} be a perverse sheaf of Λ -modules on X . Let $V \subset S$ be a dense open subscheme such that the base change $X_V \rightarrow V$ is universally locally acyclic relatively to the restriction \mathcal{F}_V of \mathcal{F} .*

Then, there exists a finite faithfully flat and generically étale morphism $S' \rightarrow S$ of regular schemes such that the base change $X' \rightarrow S'$ is locally acyclic relatively to $j'_{!} \mathcal{F}_{V'}$, where $\mathcal{F}_{V'}$ denotes the pull-back of \mathcal{F} on $V' = V \times_S S'$ and $j': X'_{V'} \rightarrow X'$ denote the base change.*

Proof. By Proposition 1.1.2.2 (1) \Rightarrow (4) and weak approximation, it suffices to consider locally on a neighborhood of each point of the complement $S - V$. Since the coefficient field Λ is finite, the assertion follows from Proposition 1.1.2.2 (3) \Rightarrow (1). \square

1.2 C -transversality

We introduce some terminology on proper intersection.

Lemma 1.2.1. *Let $f: C \rightarrow X$ and $h: W \rightarrow X$ be morphisms of schemes of finite type over a field k . Assume that C is irreducible of dimension n and that h is locally of complete intersection of relative virtual dimension d . Then every irreducible component of $h^*C = C \times_X W$ is of dimension $\geq n + d$.*

Proof. Since the assertion is local on W , we may decompose $h = gi$ as the composition of a smooth morphism g with a regular immersion of codimension c . Since the assertion is clear for g , we may assume that $h = i$ is a regular immersion. Then, it follows from [8, Proposition (5.1.7)]. \square

Definition 1.2.2. *Let $f: C \rightarrow X$ and $h: W \rightarrow X$ be morphisms of schemes of finite type over a field k . Assume that every irreducible component of C is of dimension n and that h is locally of complete intersection of relative virtual dimension d . We say that $h: W \rightarrow X$ meets $f: C \rightarrow X$ properly if $h^*C = C \times_X W$ is of dimension $n + d$.*

By Lemma 1.2.1, the condition that $h^*C = C \times_X W$ is of dimension $n + d$ is equivalent to the condition that every irreducible component of $h^*C = C \times_X W$ is of dimension $n + d$.

Lemma 1.2.3. *Let $f: C \rightarrow X$ be a morphism of schemes of finite type over a field k . Assume that X is equidimensional of dimension m and that C is equidimensional of dimension $n \geq m$. We consider the following conditions:*

(1) Every morphism $h: W \rightarrow X$ locally of complete intersection meets C properly.

(2) For every closed point x of X , the fiber $C \times_X x$ is of the dimension $n - m$.

1. We have (2) \Rightarrow (1). Assume that the condition (2) is satisfied and let $h: W \rightarrow X$ be a morphism locally of complete intersection of relative virtual dimension d of schemes of finite type over k . Then $C \times_X W$ is equidimensional of dimension $n + d$ and the morphism $C \times_X W \rightarrow W$ satisfies the condition (2) and hence (1).

2. If X is regular, we have (1) \Rightarrow (2).

3. Assume that $X = \mathbf{P}$ is a projective space and let c be an integer. Then, the linear subspaces $V \subset \mathbf{P}$ of codimension c such that the immersion $V \rightarrow \mathbf{P}$ meets C properly form a dense open subset of the Grassmannian variety \mathbf{G} .

Proof. 1. Assume that the condition (2) is satisfied and let $h: W \rightarrow X$ be a morphism locally of complete intersection of relative dimension d . Then, we have $\dim C \times_X W \leq \dim W + n - m = n + d$. Hence, $C \times_X W$ is equidimensional of dimension $n + d$ by Lemma 1.2.1. The rest is clear.

2. If X is regular and x is a closed point, the closed immersion $i: x \rightarrow X$ is a regular immersion of codimension m and hence the condition (1) implies that $\dim C \times_X x = n - m$.

3. Let $\mathbf{V} \subset \mathbf{P} \times \mathbf{G}$ be the universal family of linear subspaces of codimension c and we consider the cartesian diagram

$$\begin{array}{ccccc} & & C_{\mathbf{V}} & \longrightarrow & C \\ & \swarrow & \downarrow & \square & \downarrow \\ \mathbf{G} & \longleftarrow & \mathbf{V} & \longrightarrow & \mathbf{P}. \end{array}$$

Then, since the projection $\mathbf{V} \rightarrow \mathbf{P}$ is smooth of relative dimension $\dim \mathbf{G} - c$, we have $\dim C_{\mathbf{V}} = \dim \mathbf{G} + n - c$. Hence the open subset of \mathbf{G} consisting of V such that $\dim C \times_{\mathbf{P}} V \leq n - c$ is dense. \square

Recall that a closed subset C of a vector bundle E on a scheme X is said to be *conical* if it is stable under the action of the multiplicative group. For a closed conical subset $C \subset E$, the intersection $B = C \cap X$ with the 0-section identified with a closed subset of X is called the base of C . We say that a morphism $f: X \rightarrow Y$ of noetherian schemes is finite (resp. proper) on a closed subset $Z \subset X$ if its restriction $Z \rightarrow Y$ is finite (resp. proper) with respect to a closed subscheme structure of $Z \subset X$.

Definition 1.2.4. Let $f: X \rightarrow Y$ be a morphism of smooth schemes over a field k and let $C \subset T^*X$ be a closed conical subset.

1. ([2, 1.2]) We say that $f: X \rightarrow Y$ is C -transversal if the inverse image of C by the canonical morphism $X \times_Y T^*Y \rightarrow T^*X$ is a subset of the 0-section $X \times_Y T_Y^*Y \subset X \times_Y T^*Y$.

2. Assume that every irreducible component of X is of dimension n and that every irreducible component of C is of dimension n . Assume that every irreducible component of Y is of dimension $m \leq n$. We say that $f: X \rightarrow Y$ is properly C -transversal if $f: X \rightarrow Y$ is C -transversal and if for every closed point y of Y , the fiber $C \times_Y y$ is of dimension $n - m$.

Definition 1.2.5. Let $h: W \rightarrow X$ be a morphism of smooth schemes over a field k and let $C \subset T^*X$ be a closed conical subset. Let $K \subset W \times_X T^*X$ be the inverse image of the 0-section $T_W^*W \subset T^*W$ by the canonical morphism $W \times_X T^*X \rightarrow T^*W$.

1. ([2, 1.2]) We say that $h: W \rightarrow X$ is C -transversal if the intersection $(W \times_X C) \cap K \subset W \times_X T^*X$ is a subset of the 0-section $W \times_X T_X^*X$.

If $h: W \rightarrow X$ is C -transversal, a closed conical subset $h^\circ C \subset T^*W$ is defined to be the image of $h^*C = W \times_X C$ by $W \times_X T^*X \rightarrow T^*W$.

2. ([16, Definition 7.1]) Assume that every irreducible component of X is of dimension n and that every irreducible component of C is of dimension n . Assume that every irreducible component of W is of dimension m . We say that $h: W \rightarrow X$ is properly C -transversal if $h: W \rightarrow X$ is C -transversal and if $h: W \rightarrow X$ meets $C \rightarrow X$ properly.

If $h: W \rightarrow X$ is C -transversal, the morphism $W \times_X T^*X \rightarrow T^*W$ is finite on $h^*C = W \times_X C$ and hence $h^\circ C \subset T^*W$ is a closed subset by [2, Lemma 1.2 (ii)]. For a morphism $r: X \rightarrow Y$ of smooth schemes proper on the base $B = C \cap T_X^*X \subset X$ of a closed conical subset $C \subset T^*X$, the closed conical subset $r_\circ C \subset T^*Y$ is defined to be the image by the projection $X \times_Y T^*Y \rightarrow T^*Y$ of the inverse image of C by the canonical morphism $X \times_Y T^*Y \rightarrow T^*X$.

Lemma 1.2.6. *Let $f: X \rightarrow Y$ be a smooth morphism of smooth schemes over a field k and let $C \subset T^*X$ be a closed conical subset. Let*

$$\begin{array}{ccccc} X & \xleftarrow{h} & W & & \\ f \downarrow & \square & \downarrow g & & \\ Y & \xleftarrow{i} & Z & & \end{array}$$

be a cartesian diagram of smooth schemes over k .

1. Assume that $f: X \rightarrow Y$ is C -transversal (resp. properly C -transversal). Then, $h: W \rightarrow X$ is C -transversal (resp. properly C -transversal) and $g: W \rightarrow Z$ is $h^\circ C$ -transversal (resp. properly $h^\circ C$ -transversal).

2. Assume that $f: X \rightarrow Y$ is proper on the base of C . Then, $i: Z \rightarrow Y$ is $f_\circ C$ -transversal if and only if $h: W \rightarrow X$ is C -transversal. If these equivalent conditions are satisfied, we have $i^\circ f_\circ C = g_\circ h^\circ C$.

Proof. 1. The assertion for the transversality is proved in [16, Lemma 3.9.2]. The proper transversality of $h: W \rightarrow X$ follows from the transversality and Lemma 1.2.3 applied to $C \rightarrow Y$ and $Z \rightarrow Y$. The proper $h^\circ C$ -transversality of $g: W \rightarrow Z$ follows from that $h^*C \rightarrow h^\circ C$ is finite.

2. We consider the commutative diagram

$$\begin{array}{ccccccc} T^*X & \longleftarrow & W \times_X T^*X & \xrightarrow{dh} & T^*W & & \\ \uparrow & & \square & & \uparrow & & \\ X \times_Y T^*Y & \longleftarrow & W \times_Y T^*Y & \xrightarrow{g^*(di)} & W \times_Z T^*Z & & \\ \downarrow & & \square & & \downarrow & & \\ T^*Y & \longleftarrow & Z \times_Y T^*Y & \xrightarrow{di} & T^*Z & & \end{array}$$

with cartesian squares indicated by \square . The upper vertical arrows are injections. Since dh induces an isomorphism $W \times_X T^*X/Y \rightarrow T^*W/Z$ for the relative cotangent bundles and $f: X \rightarrow Y$ is smooth, the upper right square is also cartesian.

Let K and K' be the inverse image of the 0-sections by $dh: W \times_X T^*X \rightarrow T^*W$ and $di: Z \times_Y T^*Y \rightarrow T^*Z$ respectively. Since the upper right square is cartesian, K is identified with the inverse image of the 0-section by $g^*(di): W \times_Y T^*Y \rightarrow W \times_Z T^*Z$ which equals $g_*^{-1}(K') \subset W \times_Y T^*Y$.

Since the lower left square is cartesian, the pull-back $Z \times_Y f_*C$ is the image $g_*(C')$ of $C' = (W \times_X C) \cap (W \times_Y T^*Y)$. Hence the condition that $(Z \times_Y f_*C) \cap K' = g_*(C') \cap K' = g_*(C' \cap g_*^{-1}(K'))$ is a subset of the 0-section is equivalent to the condition that $(W \times_X C) \cap K = C' \cap g_*^{-1}(K')$ is a subset of the 0-section.

If these conditions are satisfied, the equality $i^\circ f_*C = g_*h^\circ C$ follows from the cartesian diagram. \square

Lemma 1.2.7. *Let \mathbf{P} be a projective space of dimension n and let $C \subset T^*\mathbf{P}$ be a closed conical subset.*

1. *Let \mathbf{P}^\vee be the dual projective space, let $Q \subset \mathbf{P} \times \mathbf{P}^\vee$ be the universal family of hyperplanes and let*

$$(1.4) \quad \mathbf{P} \xleftarrow{p} Q \xrightarrow{p^\vee} \mathbf{P}^\vee$$

be the projections. Let $C^\vee = p^\vee_ p^\circ C$ be the Legendre transform. Let $V \subset \mathbf{P}$ be a linear subspace and let $V^\vee \subset \mathbf{P}^\vee$ be the dual subspace. Then the immersion $V \rightarrow \mathbf{P}$ is C -transversal if and only if $V^\vee \rightarrow \mathbf{P}^\vee$ is C^\vee -transversal.*

2. *Assume that every irreducible component of C is of dimension $n = \dim \mathbf{P}$ and let $0 \leq c \leq n$ be an integer. Then, the linear subspaces $V \subset \mathbf{P}$ of codimension c such that the immersion $V \rightarrow \mathbf{P}$ is properly C -transversal form a dense open subset of the Grassmannian variety \mathbf{G} .*

Proof. 1. The C -transversality of $V \rightarrow \mathbf{P}$ means $\mathbf{P}(T_V^*\mathbf{P}) \cap \mathbf{P}(C) = \emptyset$ and similarly for the C^\vee -transversality of $V^\vee \rightarrow \mathbf{P}^\vee$. Then, the assertion follows from $\mathbf{P}(T_V^*\mathbf{P}) = \mathbf{P}(T_{V^\vee}^*\mathbf{P}^\vee)$ and $\mathbf{P}(C) = \mathbf{P}(C^\vee)$ under the identification $\mathbf{P}(T^*\mathbf{P}) = Q = \mathbf{P}(T^*\mathbf{P}^\vee)$.

2. Since the condition is an open condition on V , it suffices to show the existence. By induction on c , it is reduced to the case $c = 1$. By 1, the hyperplanes H such that the immersion $H \rightarrow \mathbf{P}$ is C -transversal is parametrized by the complement of the image $p^\vee(\mathbf{P}(C)) \subsetneq \mathbf{P}^\vee$. Hence, the assertion follows from this and Lemma 1.2.3.3. \square

1.3 $SS\mathcal{F}$ -transversality

For the definitions and basic properties of the singular support of a constructible complex on a smooth scheme over a field, we refer to [2] and [16]. Let k be a field and let Λ be a finite field of characteristic ℓ invertible in k . Let X be a smooth scheme over k such that every irreducible component is of dimension n and let \mathcal{F} be a constructible complex on X . The singular support $SS\mathcal{F}$ is defined in [2] as a closed conical subset of the cotangent bundle T^*X . By [2, Theorem 1.3 (ii)], every irreducible component C_a of the singular support

$$SS\mathcal{F} = C = \bigcup_a C_a$$

is of dimension $n = \dim X$.

Further if k is perfect, the characteristic cycle

$$CC\mathcal{F} = \sum_a m_a C_a$$

is defined as a linear combination with \mathbf{Z} -coefficients in [16, Definition 5.10]. It is characterized by the Milnor formula

$$(1.5) \quad -\dim \operatorname{tot} \phi_u(\mathcal{F}, f) = (CC\mathcal{F}, df)_{T^*U, u}$$

for morphisms $f: U \rightarrow Y$ to smooth curves Y defined on an étale neighborhood U of an isolated characteristic point u . For more detail on the notation, we refer to [16, Section 5.2].

Lemma 1.3.1. *Let $h: W \rightarrow X$ be a morphism of smooth schemes over a field k . Let \mathcal{F} be a constructible complex of Λ -modules on X and let C denote the singular support $SS\mathcal{F}$. If $h: W \rightarrow X$ is properly C -transversal, we have*

$$SSh^*\mathcal{F} = h^\circ SS\mathcal{F}.$$

Proof. By [2, Theorem 1.4 (iii)], we may assume that k is perfect. Suppose $\dim W = \dim X + d$. If \mathcal{F} is a perverse sheaf on X , then $h^*\mathcal{F}[d]$ is a perverse sheaf on W by the assumption that $h: W \rightarrow X$ is C -transversal and by [16, Lemma 8.6.5]. Hence by [2, Theorem 1.4 (ii)], we may assume that \mathcal{F} is a perverse sheaf. By [16, Proposition 5.14.2], we have $CC\mathcal{F} \geq 0$ and the support of $CC\mathcal{F}$ equals the singular support $SS\mathcal{F}$. Also we have $(-1)^d CCh^*\mathcal{F} \geq 0$ and the support of $CCh^*\mathcal{F}$ equals the singular support $SSh^*\mathcal{F}$.

By the assumption that $h: W \rightarrow X$ is properly C -transversal and by [16, Theorem 7.6], we have $CCh^*\mathcal{F} = h^!CC\mathcal{F}$. Hence by the positivity [7, Proposition 7.1 (a)], the singular support $SSh^*\mathcal{F}$ equals the support $h^\circ SS\mathcal{F}$ of $h^!CC\mathcal{F}$. \square

Lemma 1.3.2. *Let k be a field and $f: X \rightarrow Y$ be a morphism of schemes of finite type over k . Assume that Y is smooth over k . Let \mathcal{F} be a constructible complex of Λ -modules on X . Let*

$$\begin{array}{ccc} X & \xrightarrow{i} & P \\ i' \downarrow & & \downarrow g \\ P' & \xrightarrow{g'} & Y \end{array}$$

be a commutative diagram of schemes over k such that i and i' are closed immersions and the schemes P and P' are smooth over k . Let $C = SSi_\mathcal{F} \subset X \times_P T^*P \subset T^*P$ and $C' = SSi'_*\mathcal{F} \subset X \times_{P'} T^*P' \subset T^*P'$ denote the singular supports of the direct images. Then, $P \rightarrow Y$ is C -transversal (resp. properly C -transversal) if and only if $P' \rightarrow Y$ is C' -transversal (resp. properly C' -transversal).*

Proof. By factorizing $P \rightarrow Y$ as the composition of the graph $P \rightarrow P \times Y$ and the projection $P \times Y \rightarrow Y$, we may assume that $P \rightarrow Y$ is smooth. Similarly, we may assume that $P' \rightarrow Y$ is smooth. By considering the morphism $(i, i'): X \rightarrow P \times_Y P'$, we may assume that there exists a smooth morphism $P' \rightarrow P$ compatible with the immersions of X and the morphisms to Y . Since the assertion is étale local on P , we may assume that there exists a section $s: P \rightarrow P'$ compatible with the immersions of X and the morphisms to Y . Then, we have $C' = s_*C$ and the assertion follows from [16, Lemma 3.8]. \square

Lemma 1.3.2 allows us to make the following definition.

Definition 1.3.3. Let k be a field and $f: X \rightarrow Y$ be a morphism of schemes of finite type over k . Assume that Y is smooth over k . Let \mathcal{F} be a constructible complex of Λ -modules on X .

We say that $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal) if locally on X there exist a closed immersion $i: X \rightarrow P$ to a smooth scheme P over k and a morphism $g: P \rightarrow Y$ over k such that $f = g \circ i$ and that $g: P \rightarrow Y$ is C -transversal (resp. properly C -transversal) for $C = SSi_*\mathcal{F}$.

In Definition 1.3.3, we obtain an equivalent condition by requiring that g is smooth.

Let $f: X \rightarrow Y$ be a morphism of schemes of finite type over a field k such that Y is smooth over k and let \mathcal{F} be a constructible complex of Λ -modules on X . For an open subset $U \subset X$, we say $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal) on U if the restriction $U \rightarrow Y$ of f is $SS\mathcal{F}_U$ -transversal (resp. properly $SS\mathcal{F}_U$ -transversal) for the restriction \mathcal{F}_U of \mathcal{F} on U . Similarly, for an open subset $V \subset Y$, we say $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal) on V if the base change $X \times_Y V \rightarrow V$ of f is $SS\mathcal{F}_{X \times_Y V}$ -transversal (resp. properly $SS\mathcal{F}_{X \times_Y V}$ -transversal) for the restriction $\mathcal{F}_{X \times_Y V}$ of \mathcal{F} on $X \times_Y V$.

Lemma 1.3.4. Let $f: X \rightarrow Y$ be a morphism of schemes of finite type over a field k . Assume that Y is smooth over k . Let \mathcal{F} be a constructible complex of Λ -modules on X .

1. Assume that $f: X \rightarrow Y$ is smooth and that \mathcal{F} is locally constant. Then, $f: X \rightarrow Y$ is properly $SS\mathcal{F}$ -transversal.

2. Assume that $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal. Or more weakly, suppose that there exists a quasi-finite faithfully flat morphism $Y' \rightarrow Y$ of smooth schemes over k such that the base change $f': X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal for the pull-back \mathcal{F}' of \mathcal{F} on $X' = X \times_Y Y'$.

Then, $f: X \rightarrow Y$ is universally locally acyclic relatively to \mathcal{F} .

3. The following conditions are equivalent:

(1) $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal).

(2) For every integer q and for every constituent \mathcal{G} of the perverse sheaf ${}^p\mathcal{H}^q\mathcal{F}$, the morphism $f: X \rightarrow Y$ is $SS\mathcal{G}$ -transversal (resp. properly $SS\mathcal{G}$ -transversal).

4. Let k' be an extension of k . Then $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal) if and only if the base change $f': X' \rightarrow Y'$ by $\text{Spec } k' \rightarrow \text{Spec } k$ is $SS\mathcal{F}'$ -transversal (resp. properly $SS\mathcal{F}'$ -transversal) for the pull-back \mathcal{F}' on X' of \mathcal{F} .

Proof. 1. If \mathcal{F} is locally constant, then the singular support $SS\mathcal{F}$ is a subset of the 0-section T_X^*X . Hence the assertion follows.

Since the remaining assertions 2-4 are local on X , we may take a closed immersion $i: X \rightarrow P$ to a smooth scheme P over k such that f is the composition of i with a morphism $P \rightarrow Y$ over k . Replacing X and \mathcal{F} by P and $i_*\mathcal{F}$, we may assume that X is smooth over k . Set $C = SS\mathcal{F}$.

2. If $f: X \rightarrow Y$ is C -transversal, the morphism $f: X \rightarrow Y$ is universally locally acyclic relatively to \mathcal{F} by the definition of singular support. Thus under the weaker assumption, the morphism $f': X' \rightarrow Y'$ is universally locally acyclic with respect to the pull-back \mathcal{F}' . Since $Y' \rightarrow Y$ is quasi-finite and faithfully flat, the morphism $f: X \rightarrow Y$ itself is universally locally acyclic with respect to \mathcal{F} .

3. By [2, Theorem 1.4 (ii)], the singular support $SS\mathcal{F}$ equals the union of $SS\mathcal{G}$ for the constituents \mathcal{G} of the perverse sheaves ${}^p\mathcal{H}^q\mathcal{F}$ for integers q . Hence the assertion follows.

4. By [2, Theorem 1.4 (iii)], the construction of the singular support commutes with change of base fields. Hence the assertion follows. \square

Lemma 1.3.5. *Let $f: X \rightarrow Y$ be a morphism of schemes of finite type over a field k . Assume that Y is smooth over k . Let \mathcal{F} be a constructible complex of Λ -modules on X . Assume that $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal.*

1. *Assume that \mathcal{F} is a perverse sheaf. Let $V \subset Y$ be a dense open subscheme and $j: X_V = X \times_Y V \rightarrow V$ be the open immersion. Then, there is a unique isomorphism $\mathcal{F} \rightarrow j_{!*}j^*\mathcal{F}$.*

2. *There exists a dense open subscheme $V \subset Y$ such that the base change $f: X \rightarrow V$ is properly $SS\mathcal{F}$ -transversal on V .*

Proof. 1. By [3, Corollaire 1.4.25], it suffices to show that for every constituent of \mathcal{F} , its restriction on X_V is non-trivial. Let \mathcal{G} be a constituent of \mathcal{F} . By Lemma 1.3.4.3 and 2, the morphism $f: X \rightarrow Y$ is locally acyclic relatively to \mathcal{G} . Let x be a geometric point of X such that $\mathcal{G}_x \neq 0$ and let $y \rightarrow f(x)$ be a specialization for a geometric point y of V . Then, since the canonical morphism $\mathcal{G}_x \rightarrow R\Gamma(X_{(x)} \times_{Y_{(f(x))}} y, \mathcal{G})$ is an isomorphism, the restriction of \mathcal{G} on X_V is non-trivial. Thus the assertion is proved.

2. As in the proof of Lemma 1.3.4, we may assume that X is smooth over k . Set $C = SS\mathcal{F}$. There exists a dense open subset $V \subset Y$ such that for every irreducible component C_a with the reduced scheme structure of $C = \bigcup_a C_a$, the base change $C_a \times_Y V \rightarrow V$ is flat. \square

Lemma 1.3.6. *Let $f: X \rightarrow Y$ be a morphism of schemes of finite type over a field k . Assume that Y is smooth over k . Let \mathcal{F} be a constructible complex of Λ -modules on X . Let $Y' \rightarrow Y$ be a morphism of smooth schemes over k and let*

$$\begin{array}{ccc} X & \xleftarrow{h} & X' \\ f \downarrow & \square & \downarrow f' \\ Y & \xleftarrow{\quad} & Y', \end{array}$$

be a cartesian diagram. Let \mathcal{F}' denote the pull-back of \mathcal{F} on X' .

1. *We consider the following conditions:*

(1) *$f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal).*

(2) *$f': X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal (resp. properly $SS\mathcal{F}'$ -transversal).*

Then, we have (1) \Rightarrow (2). Conversely, if $Y' \rightarrow Y$ is étale surjective, we have (2) \Rightarrow (1).

2. *Assume that $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal, that \mathcal{F} is a perverse sheaf on X and that $\dim Y' = \dim Y + d$. Then $\mathcal{F}'[d]$ is a perverse sheaf on X' .*

3. *Assume that $f: X \rightarrow Y$ is smooth and is properly $SS\mathcal{F}$ -transversal. Then, we have $SS\mathcal{F}' = h^*SS\mathcal{F}$. Further if k is perfect, we have $CC\mathcal{F}' = h^!CC\mathcal{F}$.*

Proof. Since the assertions are local on X , we may take a closed immersion $i: X \rightarrow P$ to a smooth scheme P over Y . As in the proof of Lemma 1.3.4, we may assume that $f: X \rightarrow Y$ is smooth. Set $C = SS\mathcal{F}$.

1. Assume that $f: X \rightarrow Y$ is C -transversal. The pair (h, f') of morphisms is C -transversal by Lemma 1.2.6.1. Hence, $\mathcal{F}' = h^*\mathcal{F}$ is micro-supported on h^*C by [16, Lemma 4.2.4] and we have an inclusion $SS\mathcal{F}' \subset h^*C$ and f' is $SS\mathcal{F}'$ -transversal. Thus the implication (1) \Rightarrow (2) is proved for the C -transversality. The assertion on the proper C -transversality follows from this and Lemma 1.2.6.1.

Since the formation of singular support is étale local, we have (2) \Rightarrow (1) if $Y' \rightarrow Y$ is étale surjective.

2. Since $h: X' \rightarrow X$ is C -transversal by Lemma 1.2.6.1, the assertion follows from [16, Lemma 8.6.5].

3. Since $h: X' \rightarrow X$ is properly C -transversal by Lemma 1.2.6.1, the assertion for $SS\mathcal{F}'$ (resp. for $CC\mathcal{F}'$) follows from Lemma 1.3.1 (resp. [16, Theorem 7.6]). \square

Lemma 1.3.6.3 is closely related to the subject studied in [9].

Lemma 1.3.7. *Let $f: X \rightarrow Y$ be a morphism of schemes of finite type over a field k . Assume that Y is smooth over k . Let \mathcal{F} be a constructible complex of Λ -modules on X .*

1. *Let $g: Y \rightarrow Z$ be a smooth morphism of smooth schemes over k . If $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal), then the composition $gf: X \rightarrow Z$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal).*

2. *Let $h: W \rightarrow X$ be a smooth morphism of schemes of finite type over k . If $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal), then the composition $fh: W \rightarrow Y$ is $SSh^*\mathcal{F}$ -transversal (resp. properly $SSh^*\mathcal{F}$ -transversal).*

3. *Let*

$$\begin{array}{ccc} X & \xrightarrow{r} & X' \\ & \searrow f & \downarrow f' \\ & & Y \end{array}$$

be a commutative diagram of morphisms of schemes of finite type over k . Assume that $r: X \rightarrow X'$ is proper on the support of \mathcal{F} and that $f: X \rightarrow Y$ is quasi-projective. If $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal, then $f': X' \rightarrow Y$ is $SSRr_\mathcal{F}$ -transversal.*

Proof. 1. As in the proof of Lemma 1.3.4, we may assume that X is smooth over k . Set $C = SS\mathcal{F}$. Since $g: Y \rightarrow Z$ is smooth, the C -transversality of f implies that of gf by [16, Lemma 3.6.3]. The assertion on the proper C -transversality follows from this and the smoothness of g .

2. Since the question is étale local on W , we may assume that there exists a cartesian diagram

$$\begin{array}{ccccc} W & \xrightarrow{h} & X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow i & \nearrow & \\ Q & \longrightarrow & P & & \end{array}$$

of morphisms of schemes over k such that the vertical arrows are closed immersions and the horizontal arrow $Q \rightarrow P$ is a smooth morphism of smooth schemes over k . By replacing X and \mathcal{F} by P and $i_*\mathcal{F}$, we may assume that X is smooth. Since $W \times_X T^*X \rightarrow T^*W$ is an injection and $SSh^*\mathcal{F} = h^*SS\mathcal{F}$ by Lemma 1.3.1, the assertion follows.

3. Since the assertion is local on X' , we may assume that X' and Y are affine and hence X is quasi-projective. By taking a closed immersion $i': X' \rightarrow P'$ to an affine space and by factorizing $X' \rightarrow Y$ as the composition of the immersion $(i', f'): X' \rightarrow P' \times Y$ and the projection $P' \times Y \rightarrow Y$, we may assume that X' is smooth. Similarly, we take an open subscheme P of a projective space and a closed immersion $i: X \rightarrow P$. Then, by factorizing $X \rightarrow X'$ as the composition of the immersion $(i, r): X \rightarrow P \times X'$ and the projection $P \times X' \rightarrow X'$, we may also assume that X is smooth, by [16, Lemma 3.8

(2) \Rightarrow (1)]. By [2, Lemma 2.2 (ii)], we have $SSRr_*\mathcal{F} \subset r_*SS\mathcal{F}$. Hence the assertion follows from [16, Lemma 3.8 (2) \Rightarrow (1)]. \square

We give two methods to establish $SS\mathcal{F}$ -transversality.

Lemma 1.3.8. *Let $Y \rightarrow S$ be a smooth morphism of smooth schemes of finite type over a field k and let $f: X \rightarrow Y$ be a morphism of schemes of finite type over a field k . Let \mathcal{F} be a constructible complex of Λ -modules on X . Assume that the composition $X \rightarrow S$ is properly $SS\mathcal{F}$ -transversal.*

1. *Assume that k is perfect. Then, the following conditions are equivalent:*

(1) *$f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal).*

(2) *For every closed point $s \in S$, the fiber $f_s: X_s \rightarrow Y_s$ is $SS\mathcal{F}_s$ -transversal (resp. properly $SS\mathcal{F}_s$ -transversal) for the pull-back \mathcal{F}_s of \mathcal{F} on $X_s = X \times_S s$.*

2. *Assume that \mathcal{F} is a perverse sheaf on X and that $f: X \rightarrow Y$ is locally acyclic relatively to \mathcal{F} . If there exists a closed subset $Z \subset X$ quasi-finite over S such that $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal) on the complement of Z , then $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal (resp. properly $SS\mathcal{F}$ -transversal) on X .*

Proof. 1. The implication (1) \Rightarrow (2) is a special case of Lemma 1.3.6.1. We show (2) \Rightarrow (1). Since the question is local on X , we may assume that $f: X \rightarrow Y$ is smooth. Let T^*X/S and T^*Y/S denote the relative cotangent bundles and let $C = SS\mathcal{F}$. By the assumption that $X \rightarrow S$ is C -transversal, the canonical surjection $T^*X \rightarrow T^*X/S$ is finite on C by [2, Lemma 1.2 (ii)]. Hence its image $\bar{C} \subset T^*X/S$ is a closed conical subset and $C \rightarrow \bar{C}$ is finite.

The morphism $X \rightarrow Y$ is C -transversal if and only if the inverse image of $\bar{C} \subset T^*X/S$ by the canonical injection $X \times_Y T^*Y/S \rightarrow T^*X/S$ is a subset of the 0-section. This is equivalent to that for every closed point $s \in S$ and the closed immersion $i_s: X_s \rightarrow X$, the morphism $f_s: X_s \rightarrow Y_s$ is i_s^*C -transversal. Further, under the assumption that $f: X \rightarrow Y$ is C -transversal, this is properly C -transversal if and only if $f_s: X_s \rightarrow Y_s$ is properly i_s^*C -transversal for every closed point $s \in S$.

By the assumption that $X \rightarrow S$ is properly $SS\mathcal{F}$ -transversal and by Lemma 1.3.6.3, we have $SS\mathcal{F}_s = i_s^*SS\mathcal{F} = i_s^*C$ for every closed point $s \in S$. Hence the assertion is proved.

2. By Lemma 1.3.4.4, we may assume that k is perfect. By 1 and Lemma 1.3.6.2, we may assume that S is a point and further that $S = \text{Spec } k$. As in the proof of Lemma 1.3.6, we may assume that $f: X \rightarrow Y$ is smooth of relative dimension d . Let $u \in Z$. By replacing X by a neighborhood of u , we may assume $Z = \{u\}$. Set $C = SS\mathcal{F}$, $v = f(u) \in Y$ and regard $X \times_Y T^*Y$ as a closed subscheme of T^*X .

We show that $f: X \rightarrow Y$ is C -transversal, assuming that $f: X \rightarrow Y$ is locally acyclic relatively to \mathcal{F} . Namely, we show that the intersection $C' = C \cap (X \times_Y T^*Y)$ is a subset of the 0-section $X \times_Y T_Y^*Y$. By the assumption that $f: X \rightarrow Y$ is C -transversal outside u , the intersection $C' = C \cap (X \times_Y T^*Y)$ is a subset of the union $(X \times_Y T_Y^*Y) \cup (u \times_Y T^*Y)$ with the fiber at u . Let $\omega \in u \times_Y T^*Y = v \times_Y T^*Y$ be a non-zero element. After shrinking Y to a neighborhood of $v = f(u)$ if necessary, we take a smooth morphism $Y \rightarrow \mathbf{A}^1 = \text{Spec } k[t]$ such that $dt(v) = \omega$. Then, by [16, Lemma 3.6.3], the point u is at most an isolated C -characteristic point of the composition $g: X \rightarrow Y \rightarrow \mathbf{A}^1$.

Since \mathcal{F} is a perverse sheaf, the characteristic cycle $CC\mathcal{F}$ is an effective cycle and its support equals $C = SS\mathcal{F}$ by [16, Proposition 5.14]. Let dg denote the section of $X \times_Y T^*Y \subset T^*X$ defined by the function $g^*(t)$. Since the composition $X \rightarrow Y \rightarrow \mathbf{A}^1$ is

locally acyclic relatively to \mathcal{F} by [10, Corollaire 5.2.7], we have $(CC\mathcal{F}, dg)_{T^*X, u} = 0$ by the Milnor formula (1.5). Therefore by the positivity [7, Proposition 7.1 (a)], the intersection $SS\mathcal{F} \cap dg = C' \cap dg$ is empty and hence $\omega \notin C'$. Since ω is any non-zero element of $u \times_Y T^*Y$, we conclude that $C' \cap (u \times_Y T^*Y) \subset 0$ and that $f: X \rightarrow Y$ is C -transversal.

Assume further that $f: X \rightarrow Y$ is properly C -transversal outside u . Since $f: X \rightarrow Y$ is C -transversal, the morphism $T^*X \rightarrow T^*X/Y$ to the relative cotangent bundle is finite on C by [2, Lemma 1.2 (ii)] and the image $\bar{C} \subset T^*X/Y$ of C is a closed conical subset. It is sufficient to show that for every point $y \in Y$, the fiber $\bar{C} \times_Y y$ is of dimension $\leq d$. For $y \neq f(u)$, this follows from the assumption. Assume $y = f(u)$. Then, every irreducible component of $\bar{C} \times_Y y$ is either a closure of an irreducible component of $\bar{C} \times_Y y \cap (X \times_Y y - \{u\})$ or a subset of the fiber $T_u^*(X \times_Y y)$. Since $\dim T_u^*(X \times_Y y) = d$, the assertion is proved. \square

Lemma 1.3.9. *Let*

$$\begin{array}{ccccc} W & \xrightarrow{h} & X & \xrightarrow{f} & Y \\ j' \uparrow & & \square & & \uparrow j \\ U' & \xrightarrow{h_U} & U & & \end{array}$$

be a cartesian diagram of schemes of finite type over a field k . Assume that Y is smooth over k and that j is an open immersion. Let \mathcal{F}_U be a perverse sheaf of Λ -modules on U . Let $\mathcal{F} = j_{!}\mathcal{F}_U$ and let \mathcal{F}' be a perverse sheaf on W such that the restriction $\mathcal{F}'_{U'}$ on U' is isomorphic to the pull-back $h_U^*\mathcal{F}_U$. If one of the following conditions (1) and (2) below is satisfied and if $f \circ h: W \rightarrow Y$ is $SS\mathcal{F}'$ -transversal, then $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal.*

(1) *The morphism $h: W \rightarrow X$ is proper, surjective and generically finite and the composition $W \rightarrow Y$ is quasi-projective. The scheme U is smooth of dimension d over k and there exists a locally constant sheaf \mathcal{G} of Λ -modules on U such that $\mathcal{F}_U = \mathcal{G}[d]$.*

(2) *The morphism h is quasi-finite and faithfully flat. For every constituent \mathcal{G} of the perverse sheaf \mathcal{F}_U , the pull-back $h_U^*\mathcal{G}$ is also a perverse sheaf on U' .*

Proof. Assume that (1) is satisfied. Since h is surjective, the canonical morphism $\mathcal{G} \rightarrow h_{U*}h_U^*\mathcal{G}$ is an injection. Hence the constituents of the perverse sheaf $\mathcal{F}_U = \mathcal{G}[d]$ are constituents of ${}^p\mathcal{H}^0 Rh_{U*}\mathcal{F}'_{U'} = j^*{}^p\mathcal{H}^0 Rh_*\mathcal{F}'$. Consequently, the constituents of $\mathcal{F} = j_{!*}\mathcal{F}_U$ are constituents of ${}^p\mathcal{H}^0 Rh_*\mathcal{F}'$. Since $W \rightarrow Y$ is quasi-projective, the assertion follows from Lemma 1.3.7.3 and Lemma 1.3.4.3.

Assume that (2) is satisfied. By Lemma 1.3.4.3 and the perversity assumption on the pull-backs, we may assume that \mathcal{F}_U is a simple perverse sheaf. Then, by [3, Théorème 4.3.1 (ii)], there exists a locally closed immersion $i: V \rightarrow U$ of a scheme V smooth of dimension d over k and a locally constant sheaf \mathcal{G} on V such that $\mathcal{F}_U = i_{!*}\mathcal{G}[d]$. By replacing X and U by the closure of the image of $V \rightarrow X$ and V , we may assume that $V = U$. Since the assertion is étale local on X by Lemma 1.3.6.1, we may assume that h is finite and that X, W and Y are affine. Then the assertion follows from the case (1). \square

1.4 Alteration and transversality

Let $f: X \rightarrow Y$ be a morphism of smooth schemes over a field k and let $D \subset Y$ be a divisor smooth over k . In this article, we say that $f: X \rightarrow Y$ is *semi-stable* relatively to D if étale

locally on X and on Y , there exists a cartesian diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longleftarrow & D \\ \downarrow & \square & \downarrow & \square & \downarrow \\ \mathbf{A}^n & \longrightarrow & \mathbf{A}^1 & \longleftarrow & 0 \end{array}$$

where the lower left horizontal arrow $\mathbf{A}^n = \operatorname{Spec} k[t_1, \dots, t_n] \rightarrow \mathbf{A}^1 = \operatorname{Spec} k[t]$ is defined by $t \mapsto t_1 \cdots t_n$ and the lower right horizontal arrow is the inclusion of the origin $0 \in \mathbf{A}^1$. A semi-stable morphism $f: X \rightarrow Y$ is flat and the base change $f_V: X \times_Y V \rightarrow V = Y \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$ is smooth. We recall statements on the existence of alteration.

Lemma 1.4.1. *Let k be a perfect field and let $f: X \rightarrow Y$ be a dominant separated morphism of integral schemes of finite type over k .*

1. *There exists a commutative diagram*

$$(1.6) \quad \begin{array}{ccc} X & \longleftarrow & W \\ f \downarrow & & \downarrow g \\ Y & \longleftarrow & Y' \end{array}$$

of integral schemes of finite type over k satisfying the following condition: The bottom horizontal arrow $Y' \rightarrow Y$ is dominant and is the composition gh of an étale morphism g and a finite flat radicial morphism h . The schemes W and Y' are smooth over k and the morphism $g: W \rightarrow Y'$ is quasi-projective and smooth. The induced morphism $W \rightarrow X \times_Y Y'$ is proper surjective and generically finite.

2. *Let $\xi \in Y$ be a point such that the local ring $\mathcal{O}_{Y,\xi}$ is a discrete valuation ring. Then, there exists a commutative diagram (1.6) of integral schemes of finite type over k satisfying the following condition: The bottom horizontal arrow $Y' \rightarrow Y$ is quasi-finite and flat and its image is an open neighborhood of ξ . The schemes W and Y' are smooth over k , the closure $D' \subset Y'$ of the inverse image of ξ is a divisor smooth over k and the morphism $g: W \rightarrow Y'$ is quasi-projective and is semi-stable relatively to D' . The induced morphism $W \rightarrow X \times_Y Y'$ is proper surjective and generically finite.*

Proof. 1. Let η be the generic point of Y . Then, it suffices to apply [5, Theorem 4.1] to the generic fiber $X \times_Y \eta$.

2. Let $S = \operatorname{Spec} \mathcal{O}_{Y,\xi}$ be the localization at ξ . Then, it suffices to apply [5, Theorem 8.2] to the base change $X \times_Y S \rightarrow S$. \square

We prove an analogue of the generic local acyclicity theorem [6, Théorème 2.13].

Proposition 1.4.2. *Let $f: X \rightarrow Y$ be a morphism of schemes of finite type over a perfect field k . Let \mathcal{F} be a constructible complex of Λ -modules on X . Then, there exists a cartesian diagram*

$$(1.7) \quad \begin{array}{ccc} X & \longleftarrow & X' \\ f \downarrow & \square & \downarrow f' \\ Y & \longleftarrow & Y' \end{array}$$

of schemes of finite type over k satisfying the following conditions: The scheme Y' is smooth over k and the morphism $Y' \rightarrow Y$ is dominant and is the composition gh of an

étale morphism g and a finite flat radicial morphism h . The morphism $f': X' \rightarrow Y'$ is properly $SS\mathcal{F}'$ -transversal for the pull-back \mathcal{F}' of \mathcal{F} on X' .

Proof. We may assume that \mathcal{F} is a simple perverse sheaf by Lemma 1.3.4.3 and Lemma 1.3.6. Hence, we may assume that there exist a locally closed immersion $j: Z \rightarrow X$ of a smooth irreducible scheme of dimension d and a simple locally constant sheaf \mathcal{G} of Λ -modules such that $j_{!*}\mathcal{G}[d] = \mathcal{F}$ by [3, Théorème 4.3.1 (ii)]. By replacing X by the closure of $j(Z)$, we may assume that $j: Z \rightarrow X$ is an open immersion. It suffices to consider the case where $Z \rightarrow Y$ is dominant since the assertion is clear if otherwise.

Let $Z_1 \rightarrow Z$ be a finite étale covering such that the pull-back of \mathcal{G} is constant and let X_1 be the normalization of X in Z_1 . Applying Lemma 1.4.1.1 to $X_1 \rightarrow Y$, we obtain a commutative diagram

$$(1.8) \quad \begin{array}{ccc} X & \xleftarrow{r} & W \\ f \downarrow & & \downarrow \\ Y & \xleftarrow{\quad} & Y' \end{array}$$

of schemes over k satisfying the following conditions: The scheme Y' is smooth and the morphism $Y' \rightarrow Y$ is dominant and is the composition gh of an étale morphism g and a finite flat radicial surjective morphism h . The morphism $W \rightarrow Y'$ is quasi-projective and smooth. The induced morphism $r': W \rightarrow X' = X \times_Y Y'$ is proper surjective and generically finite. The pull-back \mathcal{G}'_W of \mathcal{G} on $W \times_X Z$ is a constant sheaf.

We consider the cartesian diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\quad} & Z' & \xleftarrow{\quad} & W \times_X Z \\ j \downarrow & \square & j' \downarrow & \square & \downarrow j_W \\ X & \xleftarrow{\quad} & X' & \xleftarrow{r'} & W \end{array}$$

and let \mathcal{G}' be the pull-back of \mathcal{G} on Z' . Since the finite radicial surjective morphism h is universally a homeomorphism, we have $\mathcal{F}' = j'_{!*}\mathcal{G}'[d]$.

Since \mathcal{G}'_W is a constant sheaf on $W \times_X Z$ and W is smooth over k , the intermediate extension $j_{W!}\mathcal{G}'_W[d]$ is constant. The smooth morphism $W \rightarrow Y'$ is properly $SSj_{W!}\mathcal{G}'_W[d]$ -transversal by Lemma 1.3.4.1. Since $W \rightarrow X'$ is proper and $W \rightarrow Y'$ is quasi-projective, the morphism $X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal by the case (1) in Lemma 1.3.9. After shrinking Y' , the morphism $X' \rightarrow Y'$ is properly $SS\mathcal{F}'$ -transversal by Lemma 1.3.5.2. \square

Corollary 1.4.3. *Let $f: X \rightarrow Y$ and \mathcal{F} be as in Proposition 1.4.2 and assume that k is of characteristic $p > 0$. Then, there exist a dense open subscheme $V \subset Y$ smooth over k and an iteration $\tilde{V} \rightarrow V$ of Frobenius such that the base change $X \times_Y \tilde{V} \rightarrow \tilde{V}$ is $SS\tilde{\mathcal{F}}$ -transversal for the pull-back $\tilde{\mathcal{F}}$ on $\tilde{X}_V = X \times_Y \tilde{V}$.*

Proof. After shrinking Y' in the conclusion of Proposition 1.4.2, we may assume that $Y' \rightarrow Y$ is the composition jgh of an open immersion $j: V \rightarrow Y$, a finite surjective radical morphism g and an étale surjective morphism h . By Lemma 1.3.6.1, we may assume that $Y' \rightarrow Y$ is jg . Thus, the assertion follows. \square

We show an analogue of the stable reduction theorem.

Proposition 1.4.4. *Let*

$$(1.9) \quad \begin{array}{ccc} X & \xleftarrow{\supset} & U \\ f \downarrow & \square & \downarrow f_V \\ Y & \xleftarrow{\supset} & V \end{array}$$

be a cartesian diagram of schemes of finite type over a perfect field k . Assume that Y is normal and that V is a dense open subset of Y smooth over k . Let \mathcal{F}_U be a perverse sheaf of Λ -modules on U such that $f_V: U \rightarrow V$ is $SS\mathcal{F}_U$ -transversal.

Then, there exists a cartesian diagram

$$(1.10) \quad \begin{array}{ccccccc} X & \xleftarrow{\quad} & X' & \xleftarrow{j'} & U' = U \times_X X' \\ f \downarrow & \square & \downarrow f' & \square & \downarrow \\ Y & \xleftarrow{g} & Y' & \xleftarrow{\supset} & V' = V \times_Y Y' \end{array}$$

of schemes of finite type over k satisfying the following conditions: The scheme Y' is smooth over k and $V' \subset Y'$ is the complement of a divisor $D' \subset Y'$ smooth over k . The morphism $g: Y' \rightarrow Y$ is quasi-finite flat and the complement $Y - g(Y')$ is of codimension ≥ 2 in Y . The pull-back $\mathcal{F}'_{U'}$ of \mathcal{F}_U is a perverse sheaf on U' and for $\mathcal{F}' = j'_{!}\mathcal{F}'_{U'}$ on X' , the morphism $f': X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal.*

First, we prove a basic case.

Lemma 1.4.5. *Let X and Y be smooth schemes over a field k . Let $D \subset Y$ be a divisor smooth over k and $V = Y - D$ be the complement. Let $f: X \rightarrow Y$ be a morphism over k semi-stable relatively to D . Assume that $\dim X = n$. For a cartesian diagram (1.10) such that $Y' \rightarrow Y$ is a quasi-finite flat morphism of smooth schemes over k , let \mathcal{F}' be the perverse sheaf $\mathcal{F}' = j'_{!*}\Lambda_{U'}[n]$ on X' .*

1. *Assume that $\dim Y = 1$. Let $Y' \rightarrow Y$ be a flat morphism of smooth curves over k such that for every $y' \in Y' - V'$, the action of the inertia group $I_{y'}$ on $R\Psi_{y'}\Lambda_{U'}$ is trivial. Then the morphism $X' \rightarrow Y'$ is properly $SS\mathcal{F}'$ -transversal.*

2. *There exists a quasi-finite faithfully flat morphism $Y' \rightarrow Y$ of smooth schemes over k satisfying the following conditions: The open subscheme $V' \subset Y'$ is the complement of a divisor D' smooth over k and the morphism $X' \rightarrow Y'$ is properly $SS\mathcal{F}'$ -transversal.*

Proof. 1. Since the question is étale local, we may assume that $Y = \mathbf{A}_k^1 = \text{Spec } k[t]$, that $X = X_n = \mathbf{A}_k^n = \text{Spec } k[t_1, \dots, t_n]$ and that the morphism $f: X \rightarrow Y$ is defined by $t \mapsto t_1 \cdots t_n$. We prove the assertion by induction on n . If $n = 1$, then $f: X \rightarrow Y$ is étale and \mathcal{F}' is constant. Hence the assertion follows in this case by Lemma 1.3.4.1.

Assume $n > 1$. Outside the closed point $u \in X$ defined by $t_1 = \cdots = t_n = 0$, locally there exists a smooth morphism $X = X_n \rightarrow X_{n-1}$ over Y . Hence, the induction hypothesis implies the assertion on the complement $X - \{u\}$ by Lemma 1.3.7.2. Thus, the morphism $f': X' \rightarrow Y'$ is properly $SS\mathcal{F}'$ -transversal outside the inverse image of u . By Proposition 1.1.2.2 (3) \Rightarrow (1), the morphism $f': X' \rightarrow Y'$ is locally acyclic relatively to \mathcal{F}' . Hence by Lemma 1.3.8.2, the morphism $f': X' \rightarrow Y'$ is properly $SS\mathcal{F}'$ -transversal on X' .

2. It follows from 1 by Lemma 1.3.6.1 and Lemma 1.3.5.1. \square

Proof of Proposition 1.4.4. The proof is similar to that of Proposition 1.4.2. By Lemma 1.3.6 and Lemma 1.3.5, it suffices to show the assertion on a neighborhood of each point $\xi \in Y$ of codimension 1 not contained in V . Thus, we may assume that the closure D of ξ is a divisor smooth over k and that $V = Y - D$.

We may assume that \mathcal{F}_U is a simple perverse sheaf by Lemma 1.3.4.3 and Lemma 1.3.6. Hence, similarly as in the proof of Proposition 1.4.2, we may assume that there exist a dense open immersion $j: Z \rightarrow U$ of a smooth irreducible scheme of dimension d and a simple locally constant sheaf \mathcal{G} of Λ -modules such that $\mathcal{F}_U = j_{!*}\mathcal{G}[d]$. Further, we may assume that $Z \rightarrow Y$ is dominant.

Taking a finite étale covering trivializing \mathcal{G} and applying Lemma 1.4.1.2 as in the proof of Proposition 1.4.2, we obtain a commutative diagram

$$(1.11) \quad \begin{array}{ccc} X & \xleftarrow{r} & W_1 \\ f \downarrow & & \downarrow \\ Y & \xleftarrow{g} & Y_1 \end{array}$$

of schemes over k satisfying the following conditions: The scheme Y_1 is smooth over k , the morphism $g_1: Y_1 \rightarrow Y$ is quasi-finite and flat and $Y - g_1(Y_1)$ is of codimension ≥ 2 in Y . The inverse image $V \times_Y Y_1$ is the complement of a divisor D_1 smooth over k and the morphism $W_1 \rightarrow Y_1$ is quasi-projective and is semi-stable relatively to D_1 . The induced morphism $r_1: W_1 \rightarrow X_1 = X \times_Y Y_1$ is proper surjective and generically finite. The pull-back \mathcal{G}'_1 of \mathcal{G} on $W_1 \times_X Z$ is a constant sheaf.

By Lemma 1.4.5.2 applied to the semi-stable morphism $W_1 \rightarrow Y_1$, we obtain a quasi-finite faithfully flat morphism $Y' \rightarrow Y_1$ of smooth schemes satisfying the condition loc. cit. We consider the cartesian diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\quad} & Z' & \xleftarrow{\quad} & W' \times_X Z \\ j_Z \downarrow & \square & j_{Z'} \downarrow & \square & \downarrow j_W \\ X & \xleftarrow{\quad} & X' & \xleftarrow{r'} & W' \\ & & = X \times_Y Y' & & = W_1 \times_{Y_1} Y' \end{array}$$

and let \mathcal{G}' and $\mathcal{G}'_{W'}$ denote the pull-backs of \mathcal{G} on Z' and on $W' \times_X Z$ respectively. Since $\mathcal{G}'_{W'}$ is a constant sheaf on $W' \times_X Z$, the morphism $W' \rightarrow Y'$ is $SSj_{W'*}\mathcal{G}'_{W'}[d]$ -transversal by Lemma 1.4.5.2.

The pull-back $\mathcal{F}'_{U'}$ is a perverse sheaf by Lemma 1.3.6. The perverse sheaf $\mathcal{F}' = j'_{!*}\mathcal{F}'_{U'}$ is canonically identified with $j_{Z'*}\mathcal{G}'[d]$. Since $r': W' \rightarrow X'$ is proper surjective and generically finite and since $r': W' \rightarrow Y'$ is quasi-projective, the morphism $X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal by the case (1) in Lemma 1.3.9. \square

Corollary 1.4.6. *Let the cartesian diagram (1.9) and a perverse sheaf \mathcal{F}_U on $U = X \times_Y V$ be as in Proposition 1.4.4. Assume further that Y is smooth over k and that V is the complement of a divisor $D \subset Y$ smooth over k . Then, there exist a cartesian diagram (1.10) satisfying the following conditions: The scheme Y' is smooth over k and $V' \subset Y'$ is the complement of a divisor $D' \subset Y'$ smooth over k . The morphism $g: Y' \rightarrow Y$ is quasi-finite flat, the morphism $D' \rightarrow D$ is dominant and the morphism $V' \rightarrow V$ is étale. The pull-back $\mathcal{F}'_{U'}$ of \mathcal{F}_U is a perverse sheaf on U' and the morphism $f': X' \rightarrow Y'$ is universally locally acyclic relatively to $\mathcal{F}' = j'_{!*}\mathcal{F}'$.*

Proof. Let $V' \subset Y'$ be as in the conclusion of Proposition 1.4.4. Let \bar{Y}'' be the normalization of Y in the separable closure of $k(Y)$ in $k(Y')$. Then, there exists a dense open subset $Y'' \subset \bar{Y}''$ smooth over k of the image of $Y' \rightarrow \bar{Y}''$ such that $g'': Y'' \rightarrow Y$ is flat, that $V'' = V \times_Y Y''$ is the complement of a divisor D'' smooth over k , that $D'' \rightarrow D$ is dominant, and that $V'' \rightarrow V$ is *étale*. Since $Y' \times_{\bar{Y}''} Y'' \rightarrow Y''$ is finite surjective radicial, the cartesian diagram (1.10) defined by $Y'' \rightarrow Y$ in place of $Y' \rightarrow Y$ satisfies the conditions. \square

Corollary 1.4.7. *Let $X \rightarrow Y$ be a morphism of schemes of finite type over a field k and assume that Y is smooth of dimension 1. Then, for a constructible complex \mathcal{F} of Λ -modules on X , the following conditions are equivalent:*

- (1) $X \rightarrow Y$ is locally acyclic relatively to \mathcal{F} .
- (2) $X \rightarrow Y$ is universally locally acyclic relatively to \mathcal{F} .
- (3) *There exists a finite faithfully flat morphism $Y' \rightarrow Y$ of smooth curves over k such that the base change $X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal for the base change \mathcal{F}' of \mathcal{F} on X' .*

The equivalence (1) \Leftrightarrow (2) is proved in [15].

Proof. We show (1) \Rightarrow (3). Since the nearby cycles functor is t -exact, we may assume that \mathcal{F} is a perverse sheaf. Then, the assertion follows from Propositions 1.4.2, 1.4.4 and 1.1.2. The implication (3) \Rightarrow (2) is proved in Lemma 1.3.4.2. The implication (2) \Rightarrow (1) is trivial. \square

The following example shows that taking a covering $Y' \rightarrow Y$ in condition (3) is necessary.

Example 1.4.8. Let k be a field of characteristic $p > 2$. Let $X = \mathbf{A}^1 \times \mathbf{P}^1$ and $j: U = \mathbf{A}^1 \times \mathbf{A}^1 = \text{Spec } k[x, y] \rightarrow X$ be the open immersion. Let \mathcal{G} be the locally constant sheaf of Λ -modules of rank 1 on U defined by the Artin-Schreier covering $t^p - t = xy$ and by a non-trivial character $\mathbf{F}_p \rightarrow \Lambda^\times$. Then, the second projection $\text{pr}_2: X \rightarrow Y = \mathbf{P}^1$ is locally acyclic relatively to $\mathcal{F} = j_! \mathcal{G}$ [14, Théorème 2.4.4].

On the other hand, the singular support $C = SS\mathcal{F}$ is the union of the zero-section $T_X^* X$ and the conormal bundles $T_{X_\infty}^* X$ of the fiber $X_\infty = \text{pr}_2^{-1}(\infty)$ and $T_{(0, \infty)}^* X$ of the point $(0, \infty)$. Hence the projection $\text{pr}_2: X \rightarrow Y = \mathbf{P}^1$ is not C -transversal.

Let $Y' = \mathbf{P}^1 \rightarrow Y = \mathbf{P}^1$ be the Frobenius. Then, for the pull-back \mathcal{F}' of \mathcal{F} on $X' = X \times_Y Y'$, the singular support $C' = SS\mathcal{F}'$ is the union of the zero-section $T_{X'}^* X'$ and the image of the pull-back $X'_\infty \times_{\mathbf{A}^1} T^* \mathbf{A}^1 \rightarrow T^* X'$ with respect to the first projection on the fiber $X'_\infty = \text{pr}_2^{-1}(\infty)$ at infinity. Consequently, the projection $\text{pr}_2: X' \rightarrow Y' = \mathbf{P}^1$ is C' -transversal.

Let $Y'' \rightarrow Y'$ be a flat morphism of smooth curves over k and let \mathcal{F}'' be the pull-back of \mathcal{F}' on $X'' = X' \times_{Y'} Y''$. Then, the morphism $h: X'' \rightarrow X'$ is properly C' -transversal and hence $SS\mathcal{F}'' = h^* SS\mathcal{F}'$ is the union of $T_{X''}^* X''$ and $X''_\infty \times_{\mathbf{A}^1} T^* \mathbf{A}^1 \subset T^* X''$ by Lemma 1.3.1.

1.5 Potential transversality

We prove a refinement of the analogue of the stable reduction theorem, using the following consequence of the stable reduction theorem for curves.

Lemma 1.5.1. *Let*

$$\begin{array}{ccccc} U & \xrightarrow{\subset} & X & & \\ f_V \downarrow & \square & \downarrow f & & \\ V & \xrightarrow{\subset} & Y & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of morphisms of smooth schemes of finite type over a perfect field k satisfying the following conditions: The morphism $f: X \rightarrow Y$ is flat and the morphisms $g: Y \rightarrow S$ and $f_V: U \rightarrow V$ are smooth of relative dimension 1. The horizontal arrows are open immersions and the open subset $V \subset Y$ is the complement of a divisor $D \subset Y$ smooth over k and quasi-finite and flat over S .

Then, there exists a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & S' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & S \end{array}$$

of smooth schemes over k satisfying the following conditions: The morphisms $S' \rightarrow S$, $Y' \rightarrow Y \times_S S'$ and $X' \rightarrow X \times_Y Y'$ are quasi-finite flat and dominant. The morphisms $g': Y' \rightarrow S'$ and $f': X' \rightarrow Y'$ are smooth of relative dimension 1 and that $V' = V \times_Y Y' \subset Y'$ is the complement of a divisor $D' \subset Y'$ smooth over k and quasi-finite and flat over S' . The morphism $V' \rightarrow V \times_S S'$ is étale and the morphism $U' = X' \times_{Y'} V' \rightarrow U \times_V V'$ is an isomorphism. The quasi-finite morphisms $D' \rightarrow D$ and $X' \times_{Y'} D' \rightarrow X \times_Y D'$ are dominant.

Proof. Let $\bar{\eta}$ be a geometric point of S defined by an algebraic closure of the function field of an irreducible component. Then, it suffices to apply [18, Theorem 1.5] to the base change of $X \rightarrow Y \rightarrow S$ by $\bar{\eta} \rightarrow S$. \square

Theorem 1.5.2. *Let*

$$\begin{array}{ccccc} U & \xrightarrow{\subset} & X & & \\ \downarrow & \square & \downarrow f & & \\ V & \xrightarrow{\subset} & Y & \longrightarrow & S \end{array}$$

be a cartesian diagram of morphisms of schemes of finite type over a perfect field k . Assume that Y and S are smooth over k , that $Y \rightarrow S$ is smooth of relative dimension 1 and that $V \subset Y$ is the complement of a divisor D smooth over k and quasi-finite and flat over S . Let \mathcal{F}_U be a perverse sheaf of Λ -modules on $U = X \times_Y V$ such that $U \rightarrow V$ is $SS\mathcal{F}_U$ -transversal.

Then, there exists a commutative diagram

$$(1.12) \quad \begin{array}{ccccc} V' & \xrightarrow{\subset} & Y' & \longrightarrow & S' \\ \downarrow & \square & \downarrow & & \downarrow \\ V & \xrightarrow{\subset} & Y & \longrightarrow & S \end{array}$$

of smooth schemes over k satisfying the following conditions (1) and (2):

(1) *The morphisms $S' \rightarrow S$ and $Y' \rightarrow Y \times_S S'$ are quasi-finite flat and dominant. The horizontal arrow $Y' \rightarrow S'$ is smooth of relative dimension 1. The left square is cartesian*

and $V' \subset Y'$ is the complement of a divisor $D' \subset Y'$ smooth over k and quasi-finite and flat over S' . The induced morphism $V' \rightarrow V \times_S S'$ is étale and $D' \rightarrow D$ is dominant.

(2) Let

$$(1.13) \quad \begin{array}{ccccc} U' & \xrightarrow{j'} & X' & \xrightarrow{f'} & Y' \\ \downarrow & \square & \downarrow & \square & \downarrow \\ U & \longrightarrow & X & \xrightarrow{f} & Y' \end{array}$$

be a cartesian diagram and let $\mathcal{F}'_{U'}$ denote the pull-back of \mathcal{F}_U on U' . Then the morphism $f': X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal for $\mathcal{F}' = j'_{!*}\mathcal{F}'_{U'}$.

Proof. Since the assertion is local on X , we may assume that there exists a closed immersion $i: X \rightarrow \mathbf{A}_Y^n$ for an integer $n \geq 0$. By replacing X and \mathcal{F}_U by \mathbf{A}_Y^n and $i|_{U*}\mathcal{F}_U$ on \mathbf{A}_Y^n , we may assume that X is an open subscheme of \mathbf{A}_Y^n . We prove the assertion by induction on n .

Assume $n = 0$ and hence $X \rightarrow Y$ is an open immersion. Since the open immersion $U \rightarrow V$ is $SS\mathcal{F}_U$ -transversal, the singular support $SS\mathcal{F}_U$ is a subset of the 0-section T_U^*U by Lemma [16, Lemma 3.6.3]. Hence \mathcal{F}_U is locally constant by [2, Lemma 2.1(iii)]. Let $U_1 \rightarrow U$ be a finite étale covering such that the pull-back of \mathcal{F}_U is constant. Let Y_1 be the normalization of Y in U_1 . There exists a quasi-finite flat and dominant morphism $S' \rightarrow S$ of smooth scheme such that the normalization Y' of $Y_1 \times_S S'$ is smooth over S' and that $V' \subset Y'$ is the complement of a divisor D' étale over S' . After shrinking S' , we may assume that $Y' \rightarrow Y \times_S S'$ is flat. After shrinking Y' keeping D' dominant over D , we may assume that $V' \rightarrow V \times_S S'$ étale. Then, the condition (1) is satisfied. Since \mathcal{F}' on $X' \subset Y'$ is constant, the condition (2) is also satisfied by Lemma 1.3.4.1.

Assume that $n \geq 1$ and that the assertion holds for $n-1$. For the proof of the induction step, we first show the following weaker assertion.

Claim. Let $X \subset \mathbf{A}_Y^n \rightarrow \mathbf{A}_Y^1$ be a projection and assume that its restriction $U \subset \mathbf{A}_Y^n \rightarrow \mathbf{A}_Y^1$ is $SS\mathcal{F}_U$ -transversal. Then, there exist a commutative diagram (1.12) satisfying the condition (1) and an open subset $W' \subset \mathbf{A}_{D'}^1$, satisfying the following condition:

(2') The intersection $W' \cap \mathbf{A}_{D'}^1$ is dense in $\mathbf{A}_{D'}^1$. For the cartesian diagram (1.13) and for the pull-back $\mathcal{F}'_{U'}$ of \mathcal{F} on U' and $\mathcal{F}' = j'_{!*}\mathcal{F}'_{U'}$ on X' , the morphism $f': X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal on the inverse image $X' \times_{\mathbf{A}_{Y'}^1} W' \subset X'$.

Proof of Claim. By the induction hypothesis applied to $X \subset \mathbf{A}_Y^n \rightarrow \mathbf{A}_Y^1 \rightarrow \mathbf{A}_S^1$, there exists a commutative diagram

$$\begin{array}{ccc} Y_1 & \longrightarrow & S_1 \\ \downarrow & & \downarrow \\ \mathbf{A}_Y^1 & \longrightarrow & \mathbf{A}_S^1 \end{array}$$

satisfying the conditions (1) and (2) in Theorem 1.5.2. We consider the cartesian diagram

$$\begin{array}{ccccc} U_1 & \xrightarrow{j_1} & X_1 & \longrightarrow & Y_1 \\ \downarrow & \square & \downarrow & \square & \downarrow \\ U & \longrightarrow & X & \longrightarrow & \mathbf{A}_Y^1 \end{array}$$

and let \mathcal{F}_{U_1} be the pull-back of \mathcal{F}_U . Then, for $\mathcal{F}_1 = j_{1!}\mathcal{F}_{U_1}$ on X_1 , the morphism $X_1 \rightarrow Y_1$ is $SS\mathcal{F}_1$ -transversal. The inverse image $V_1 = V \times_Y Y_1 \subset Y_1$ is the complement of a divisor $D_1 \subset Y_1$ smooth over k and quasi-finite and flat over S_1 . The quasi-finite morphism $V_1 \rightarrow V \times_S S_1$ is *étale* and the quasi-finite morphism $D_1 \rightarrow \mathbf{A}_D^1$ is dominant.

Since the morphism $S_1 \rightarrow \mathbf{A}_S^1$ is quasi-finite and flat, there exists a quasi-finite, flat and dominant morphism $S' \rightarrow S$ of smooth schemes over k such that the normalization S'_1 of $S_1 \times_S S'$ is smooth over S' and that the induced morphism $S'_1 \rightarrow S_1$ is also quasi-finite, flat and dominant. After shrinking S'_1 if necessary, we may assume that the morphism $Y_1 \times_{S_1} S'_1 \rightarrow \mathbf{A}_Y^1 \times_{\mathbf{A}_S^1} S'_1$ of smooth curves over S'_1 is flat. Hence, by replacing S, Y, S_1 and Y_1 by $S', Y \times_S S', S'_1$ and $Y_1 \times_{S_1} S'_1$, we may assume that $S_1 \rightarrow S$ is smooth of relative dimension 1.

We consider the commutative diagram

$$(1.14) \quad \begin{array}{ccccc} V_1 & \longrightarrow & Y_1 & \longrightarrow & S_1 \\ \downarrow & \square & \downarrow & & \downarrow \\ V & \longrightarrow & Y & \longrightarrow & S \end{array}$$

where the left square is cartesian. Since $V_1 \rightarrow V \times_S S_1$ is *étale*, the left vertical arrow $V_1 \rightarrow V$ is also smooth of relative dimension 1. The middle vertical arrow $Y_1 \rightarrow Y$ is flat. Hence, by Lemma 1.5.1 applied to (1.14), there exists a commutative diagram

$$\begin{array}{ccccc} Y'_1 & \longrightarrow & Y' & \longrightarrow & S' \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y & \longrightarrow & S \end{array}$$

of smooth schemes over k satisfying the following conditions: The morphisms $S' \rightarrow S$, $Y' \rightarrow Y \times_S S'$ and $Y'_1 \rightarrow Y_1 \times_Y Y'$ are quasi-finite flat and dominant. The morphisms $Y' \rightarrow S'$ and $Y'_1 \rightarrow Y'$ are smooth of relative dimension 1. The inverse image $V' = V \times_Y Y'$ is the complement $Y' - D'$ of a divisor $D' \subset Y'$ smooth over k and quasi-finite and flat over S' . The morphism $V' \rightarrow V \times_S S'$ is *étale* and the morphism $Y'_1 \times_{Y'} V' \rightarrow V_1 \times_V V'$ is an isomorphism. The quasi-finite morphisms $D' \rightarrow D$ and $Y'_1 \times_{Y'} D' \rightarrow Y_1 \times_Y D'$ are dominant. Thus the condition (1) in Theorem 1.5.2 is satisfied.

The composition $Y'_1 \rightarrow Y_1 \times_Y Y' \rightarrow \mathbf{A}_{Y'}^1$, is quasi-finite and flat. We consider the cartesian diagram

$$\begin{array}{ccccc} X'' & \longrightarrow & Y'_1 & & \\ \downarrow & \square & \downarrow & & \\ X' & \longrightarrow & \mathbf{A}_{Y'}^1 & \longrightarrow & Y' \end{array}$$

and the pull-back \mathcal{F}'' of \mathcal{F}_1 on X'' . Then, since $X_1 \rightarrow Y_1$ is $SS\mathcal{F}_1$ -transversal, the morphism $X'' \rightarrow Y'_1$ is $SS\mathcal{F}''$ -transversal by Lemma 1.3.6.1. Since $Y'_1 \rightarrow Y'$ is smooth, the composition $X'' \rightarrow Y'$ is also $SS\mathcal{F}''$ -transversal by Lemma 1.3.7.1. By Lemma 1.3.6.2 and Lemma 1.3.4.3, the pull-back on $U'' = U' \times_{X'} X''$ of every constituent \mathcal{G} of $\mathcal{F}_{U'}'$, is a perverse sheaf. Since $Y'_1 \rightarrow \mathbf{A}_{Y'}^1$, is quasi-finite and flat, the morphism $f': X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal on the image of X'' by the case (2) in Lemma 1.3.9.

Let $W' \subset \mathbf{A}_{Y'}^1$, be the image of Y'_1 . The image of $X'' \rightarrow X'$ equals the inverse image $X' \times_{\mathbf{A}_{Y'}^1} W'$. Hence, $f': X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal on $X' \times_{\mathbf{A}_{Y'}^1} W' \subset X'$. Since

$Y'_1 \times_{Y'} D' \rightarrow Y_1 \times_Y D'$ and $D_1 \rightarrow \mathbf{A}_D^1$ are dominant, the intersection $W' \cap \mathbf{A}_{D'}^1$ is dense in $\mathbf{A}_{D'}^1$. Thus the condition (2') in Claim is also satisfied. \square

To complete the proof of the induction step, we use the following elementary lemma.

Lemma 1.5.3. *Let X be an open subset of a vector space V of dimension n over an infinite field k regarded as a smooth scheme over k . Let $C \subset T^*X$ be a closed conical subset of dimension $\leq n$. Then, there exists an isomorphism $V \rightarrow \mathbf{A}^n$ of vector spaces over k such that the compositions $X \rightarrow V \rightarrow \mathbf{A}^n \rightarrow \mathbf{A}^1$ with the projections $\text{pr}_i, i = 1, \dots, n$ have at most isolated C -characteristic points.*

Proof. Identify the cotangent bundle T^*X with the product $X \times V^\vee$ with the dual and let $\mathbf{P}(C) \subset \mathbf{P}(T^*X) = X \times \mathbf{P}(V^\vee)$ be the projectivization. Then, by the assumption $\dim C \leq n$, the projection $\mathbf{P}(C) \rightarrow \mathbf{P}(V^\vee)$ is generically finite. By the assumption that k is infinite, there exists a basis p_1, \dots, p_n of V^\vee such that the fibers of $\mathbf{P}(C) \rightarrow \mathbf{P}(V^\vee)$ at $\bar{p}_1, \dots, \bar{p}_n \in \mathbf{P}(V^\vee)$ are finite. Then, the product of $p_1, \dots, p_n: V \rightarrow \mathbf{A}^1$ satisfies the condition. \square

Set $C_U = SS\mathcal{F}_U \subset T^*U$. By the assumption that $U \subset \mathbf{A}_V^n \rightarrow V$ is $SS\mathcal{F}_U$ -transversal, the morphism $T^*U \rightarrow T^*U/V$ to the relative cotangent bundle is finite on C_U by [2, Lemma 1.2 (ii)]. The image $\bar{C}_U \subset T^*U/V$ of C_U and its closure $\bar{C} \subset T^*X/Y$ are closed conical subsets. Since every irreducible component of C_U is of dimension $\dim X$, every irreducible component of \bar{C}_U is also of dimension $\dim X$. Hence, for the generic point of each irreducible component of Y , the fiber of \bar{C}_U is of dimension $\leq n = \dim X - \dim Y$. Consequently, for the generic point of each irreducible component of $D \subset Y$, the fiber of \bar{C} is also of dimension $\leq n$.

By Lemma 1.5.3 applied to the fibers of the generic points of irreducible components of D , after replacing S by a dense open subset, there exists a coordinate of $\mathbf{A}_Y^n \supset X$ such that, for each $i = 1, \dots, n$, there exist a dense open subset $W_i \subset \mathbf{A}_D^1$ and an open neighborhood $X_i \subset X$ of the inverse image $W_i \times_{\mathbf{A}_Y^1} X$ by the i -th projection pr_i satisfying the following condition: The inverse image of $\bar{C} \subset T^*X/Y$ by the morphism $X \times_{\mathbf{A}_Y^1} T^*\mathbf{A}_Y^1/Y \rightarrow T^*X/Y$ of the relative cotangent bundles induced by pr_i is a subset of the 0-section on X_i .

Then, the restriction $U \rightarrow \mathbf{A}_Y^1$ of pr_i is $SS\mathcal{F}_U$ -transversal on $X_i \cap U$. By Claim applied to the restriction $X_i \rightarrow \mathbf{A}_Y^1$ of pr_i , there exist a commutative diagram (1.12) satisfying the condition (1) and for each $i = 1, \dots, n$ a dense open subset $W'_i \subset \mathbf{A}_{Y'}^1$, satisfying the condition (2'). Hence $X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal on the union $W' = \bigcup_{i=1}^n \text{pr}_i^{-1} W'_i \subset X' \subset \mathbf{A}_{Y'}^n$, of the inverse images by the projections. Since $X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal on U' , it is $SS\mathcal{F}'$ -transversal on $W' \cup U'$. By shrinking S' if necessary, we may assume that $Z' = X' - (W' \cup U') = \prod_{i=1}^n (\mathbf{A}_{D'}^1 - (\mathbf{A}_{D'}^1 \cap W'_i)) \subset \mathbf{A}_{D'}^n$ is quasi-finite over S' .

By Corollary 1.4.6, there exists a cartesian diagram

$$\begin{array}{ccccccc} U'' & \xrightarrow{j''} & X'' & \xrightarrow{f''} & Y'' & \longleftarrow & V'' \\ \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow \\ U' & \xrightarrow{j'} & X' & \xrightarrow{f'} & Y' & \longleftarrow & V' \end{array}$$

of smooth schemes over k satisfying the following condition: The morphism $V'' \rightarrow V'$ is étale and $V'' \subset Y''$ is the complement of a divisor $D'' \subset Y''$ smooth over k . The morphism $D'' \rightarrow D'$ is dominant. For the pull-back $\mathcal{F}_{U''}''$ of $\mathcal{F}_{U'}'$ on U'' and $\mathcal{F}'' = j_{!*}'' \mathcal{F}''$, the morphism $f'': X'' \rightarrow Y''$ is universally locally acyclic relatively to \mathcal{F}'' .

By Lemma 1.3.6.1 and Lemma 1.3.5.1, \mathcal{F}'' is the pull-back of \mathcal{F}' outside the inverse image Z'' of Z' and $f'': X'' \rightarrow Y''$ is $SS\mathcal{F}''$ -transversal outside the inverse image Z'' .

Let $S'' \rightarrow S'$ be a quasi-finite flat dominant morphism of smooth schemes over k such that the normalization Y''' of $Y'' \times_{S'} S''$ is smooth over S'' of relative dimension 1 and that $V''' = V'' \times_{Y''} Y'''$ is the complement of a divisor D''' smooth over k . Since $V'' \rightarrow V'$ is étale, the morphism $V'' \rightarrow S'$ is smooth and $V''' \rightarrow V'' \times_{S'} S''$ is an isomorphism. Hence the morphism $V''' \rightarrow V' \times_{S'} S''$ is étale. The morphism $D''' \rightarrow D'$ is dominant. We consider the commutative diagram

$$\begin{array}{ccccccc} U''' & \xrightarrow{j'''} & X''' & \xrightarrow{f'''} & Y''' & \longrightarrow & S'' \\ \downarrow & \square & \downarrow & \square & \downarrow & & \downarrow \\ U' & \xrightarrow{j'} & X' & \xrightarrow{f'} & Y' & \longrightarrow & S' \end{array}$$

where the left and middle squares are cartesian. Then for the pull-back \mathcal{F}''' of \mathcal{F}'' on X''' , the morphism $f''': X''' \rightarrow Y'''$ is universally locally acyclic and is $SS\mathcal{F}'''$ -transversal outside the inverse image Z''' of Z'' quasi-finite over S'' .

Shrinking S'' , we may further assume that $X''' \rightarrow S''$ is properly $SS\mathcal{F}'''$ -transversal by Lemma 1.3.5.2. Then, the morphism $f''': X''' \rightarrow Y'''$ is $SS\mathcal{F}'''$ -transversal by Lemma 1.3.8.2. Further we have an isomorphism $\mathcal{F}''' = j_{!*} j'''^* \mathcal{F}'''$ by Lemma 1.3.5.1. Thus, the commutative diagram

$$\begin{array}{ccccc} V''' & \xrightarrow{\subset} & Y''' & \longrightarrow & S'' \\ \downarrow & \square & \downarrow & & \downarrow \\ V & \xrightarrow{\subset} & Y & \longrightarrow & S \end{array}$$

satisfies the required conditions. \square

Corollary 1.5.4. *Let $f: X \rightarrow Y$ be a morphism of scheme of finite type over a perfect field k . Assume that Y is smooth of dimension 1. Let \mathcal{F} be a constructible complex of Λ -modules on X . Assume that $f: X \rightarrow Y$ is locally acyclic relatively to \mathcal{F} and that there exists a dense open subset $V \subset Y$ such that $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal on V . Then, there exists a cartesian diagram*

$$\begin{array}{ccc} X & \longleftarrow & X' \\ f \downarrow & \square & \downarrow f' \\ Y & \longleftarrow & Y' \end{array}$$

of morphisms of schemes of finite type over k satisfying the following condition: The morphism $Y' \rightarrow Y$ is a finite generically étale morphism of smooth curves. For the pull-back \mathcal{F}' of \mathcal{F} on X' , the morphism $f': X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal.

Proof. Since the shifted vanishing cycles functor $R\Phi[-1]$ is t -exact, we may assume that \mathcal{F} is a perverse sheaf. Then the assertion follows from the case where $S = \text{Spec } k$ in Theorem 1.5.2, Proposition 1.1.2.1 and weak approximation. \square

2 Characteristic cycles and the direct image

2.1 Direct image of a cycle

To state the compatibility with push-forward, we fix some terminology and notations. Recall that a morphism $f: X \rightarrow Y$ of noetherian schemes is said to be proper on a closed subset $Z \subset X$ if its restriction $Z \rightarrow Y$ is proper with respect to a closed subscheme structure of $Z \subset X$.

Let $f: X \rightarrow Y$ be a morphism of smooth schemes over a field k and we consider the diagram

$$(2.1) \quad T^*X \longleftarrow X \times_Y T^*Y \longrightarrow T^*Y$$

as an algebraic correspondence from T^*X to T^*Y . Assume that every irreducible component of X (resp. of Y) is of dimension n (resp. m). Let $B \subset X$ be a closed subset on which $f: X \rightarrow Y$ is proper and let $C \subset T^*X$ be a closed subset of $B \times_X T^*X$. Then, the closed subset $f_*C \subset T^*Y$ is defined as the image by the right arrow in (2.1) of the inverse image of C by the left arrow. It is a closed subset by the assumption that f is proper on B . The composition of the Gysin map [7, 6.6] for the first arrow and the push-forward map for the second arrow defines a morphism

$$(2.2) \quad f_!: \mathrm{CH}_n(C) \longrightarrow \mathrm{CH}_m(f_*C)$$

since $\dim T^*X - \dim X \times_Y T^*Y = n - m$. If every irreducible component of C (resp. f_*C) is of dimension $\leq n$ (resp. $\leq m$), the morphism (2.2) defines a morphism

$$(2.3) \quad f_!: Z_n(C) \longrightarrow Z_m(f_*C)$$

of free abelian groups of cycles.

Lemma 2.1.1. *Let*

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ & \searrow f & \downarrow f' \\ & & Y \end{array}$$

*be a commutative diagram of morphisms of smooth schemes over k . Assume that every irreducible component of X (resp. of X' and Y) is of dimension n (resp. n' and m). Let $B \subset X$ be a closed subset on which $f: X \rightarrow Y$ is proper and let $C \subset T^*X$ be a closed subset of $B \times_X T^*X$. Then, the diagram*

$$\begin{array}{ccc} \mathrm{CH}_n(C) & \xrightarrow{g_!} & \mathrm{CH}_{n'}(g_*C) \\ & \searrow f_! & \downarrow f'_! \\ & & \mathrm{CH}_m(f_*C) \end{array}$$

is commutative.

Proof. We consider the diagram

$$\begin{array}{ccccc} T^*X & \longleftarrow & X \times_{X'} T^*X' & \longrightarrow & T^*X' \\ & & \uparrow & \square & \uparrow \\ & & X \times_Y T^*Y & \longrightarrow & X' \times_Y T^*Y \longrightarrow T^*Y \end{array}$$

with cartesian square. After decomposing the right vertical arrow into the composition of a smooth morphism and a regular immersion, it suffices to apply [7, Theorem 6.2 (a)]. \square

Lemma 2.1.2. *Let $f: X \rightarrow Y$ be a smooth morphism of smooth irreducible schemes over a perfect field k . Assume that X (resp. of Y) is of dimension n (resp. m). Let $C = \bigcup_a C_a \subset T^*X$ be a closed conical subset such that every irreducible component C_a is of dimension n and that $f: X \rightarrow Y$ is properly C -transversal and is proper on the base $B = C \cap T_X^*X \subset X$.*

Let $A = \sum_a m_a C_a$ be a linear combination. Let $y \in Y$ be a closed point, let $A_y = i_y^! A$ be the pull-back [16, Definition 7.1] by the closed immersion $i_y: X_y \rightarrow X$ of the fiber and let $(A_y, T_{X_y}^ X_y)_{T^* X_y}$ denote the intersection number. Then, we have*

$$(2.4) \quad f_! A = (-1)^m (A_y, T_{X_y}^* X_y)_{T^* X_y} \cdot [T_Y^* Y]$$

in $Z_m(T_Y^ Y)$.*

Proof. Since the closed immersion $i_y: X_y \rightarrow X$ is properly C -transversal by Lemma 1.2.6.1, the pull-back $A_y = i_y^! A$ is defined. Further by the assumption that $f: X \rightarrow Y$ is C -transversal, we have an inclusion $f_* C \subset T_Y^* Y$ and $f_! A$ is defined as an element of $\text{CH}_m(f_* C) = Z_m(f_* C) \subset Z_m(T_Y^* Y)$. Hence it suffices to show that the coefficient of $T_Y^* Y$ in $f_! A$ equals the intersection number $(-1)^m (A_y, T_{X_y}^* X_y)_{T^* X_y}$.

We consider the cartesian diagram

$$\begin{array}{ccccc} T^*X & \longleftarrow & X \times_Y T^*Y & \longrightarrow & T^*Y \\ \uparrow & \square & \uparrow & \square & \uparrow \\ X_y \times_X T^*X & \longleftarrow & X_y \times_Y T^*Y & \longrightarrow & y \times_Y T^*Y \\ \downarrow & \square & \downarrow & \square & \downarrow \\ T^*X_y & \xleftarrow{0} & X_y & \longrightarrow & y. \end{array}$$

We regard the four sides of the exterior square of the diagram as algebraic correspondences. The coefficient of $f_! A$ is the image of A by the composition via the upper right corner. It equals the composition via the upper right corner by [7, Theorem 6.2 (a)] applied to the upper right and the lower left squares. Since the definition of $i_y^! A$ in [16, Definition 7.1] involves the sign $(-1)^{\dim X - \dim X_y} = (-1)^m$, the assertion follows. \square

We study the case where Y is a smooth curve and $\dim f_* C = 1$. Let $f: X \rightarrow Y$ be a morphism of smooth schemes over k . Assume that every irreducible component of X (resp. of Y) is of dimension n (resp. 1). Let $C \subset T^*X$ be a closed conical subset such that every irreducible component C_a of $C = \bigcup_a C_a$ is of dimension n and that $f: X \rightarrow Y$ is proper on the base $B = C \cap T_X^*X \subset X$. Let $V \subset Y$ be a dense open subscheme such that the base change $f_V: X_V \rightarrow V$ is properly C_V -transversal for the restriction C_V of C on X_V .

Let $y \in Y - V$ be a closed point on the boundary and let t be a uniformizer at y and let df denote the section of T^*X defined on a neighborhood of the fiber X_y by the pull-back $f^* dt$. Then, on a neighborhood of X_y , the intersection $C \cap df \subset T^*X$ is supported on the inverse image of the intersection $B \cap X_y$. Hence for a linear combination $A = \sum_a m_a C_a$, the intersection product

$$(2.5) \quad (A, df)_{T^*X, X_y}$$

supported on the fiber X_y is defined as an element of $\mathrm{CH}_0(B \cap X_y)$. Since C is conical, the intersection product $(A, df)_{T^*X, X_y}$ does not depend on the choice of t . Thus the intersection number also denoted $(A, df)_{T^*X, X_y}$ is defined as its image by the degree mapping $\mathrm{CH}_0(B \cap X_y) \rightarrow \mathrm{CH}_0(y) = \mathbf{Z}$.

Lemma 2.1.3. *Let $f: X \rightarrow Y$ be a morphism of smooth irreducible schemes over a perfect field k . Assume that X (resp. of Y) is of dimension n (resp. 1). Let $C = \bigcup_a C_a \subset T^*X$ be a closed conical subset as in Lemma 2.1.2.*

1. *The following conditions are equivalent:*

(1) $\dim f_*C \leq 1$.

(2) *There exists a dense open subscheme $V \subset Y$ such that the base change $f_V: X_V \rightarrow V$ is C_V -transversal for the restriction C_V of C on X_V .*

(3) *There exists a dense open subscheme $V \subset Y$ such that the base change $f_V: X_V \rightarrow V$ is properly C_V -transversal for the restriction C_V of C on X_V .*

2. *Let $V \subset Y$ be a dense open subscheme satisfying the condition (3) above. Let $A = \sum_a m_a C_a$ be a linear combination, let $v \in V$ be a closed point and define the intersection number $(A_v, T_{X_v}^* X_v)_{T^*X_v}$ as in Lemma 2.1.2. Then, we have*

$$(2.6) \quad f_! A = -(A_v, T_{X_v}^* X_v)_{T^*X_v} \cdot [T_Y^* Y] + \sum_{y \in Y - V} (A, df)_{T^*X, X_y} \cdot [T_y^* Y]$$

in $Z_1(f_*C)$.

Proof. 1. Since f_*C is a closed conical subset of the line bundle T^*Y , the condition (1) is equivalent to the existence of a dense open subset $V \subset Y$ such that $f_*C \subset T_Y^* Y \cup \bigcup_{y \in Y - V} T_y^* Y$. This is equivalent to the condition (2). The equivalence (2) \Leftrightarrow (3) follows from Lemma 1.3.5.2.

2. It suffices to compare the coefficients of the 0-section $T_Y^* Y$ and of the fibers $T_y^* Y$ respectively. For those of $T_Y^* Y$, it is proved in Lemma 2.1.2. For those of $T_y^* Y$, it follows from the projection formula [7, Theorem 6.2 (a)] applied to the cartesian square in the diagram

$$\begin{array}{ccccc} T^*X & \longleftarrow & X \times_Y T^*Y & \longrightarrow & T^*Y \\ & & \uparrow df & \square & \uparrow dt \\ & & X & \longrightarrow & Y. \end{array}$$

□

Lemma 2.1.4. *Let X be a scheme of finite type of dimension d over a field k and let E be a vector bundle on X associated to a locally free \mathcal{O}_X -module \mathcal{E} of rank n . Let $s: X \rightarrow E$ be a section, $0: X \rightarrow E$ be the zero section and $Z = Z(s) = 0(X) \cap s(X) \subset X$ be the zero locus of s . Let $\mathcal{K} = [\mathcal{O}_X \xrightarrow{s} \mathcal{E}]$ be the complex of \mathcal{O}_X -modules where \mathcal{E} is put on degree 0 and let $c_{nZ}^X(\mathcal{K})$ be the localized Chern class defined in [4, Section 1]. Then, we have*

$$(0(X), s(X))_E = c_{nZ}^X(\mathcal{K}) \cap [X]$$

in $\mathrm{CH}_{d-n}(Z)$.

Proof. We may assume that X is integral and $Z \subsetneq X$. By taking the blow-up at Z and by [13, Proposition 2.3.1.6], we may assume that Z is a Cartier divisor $D \subset X$. Then, we

have an exact sequence of $0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$ of locally free \mathcal{O}_X -modules where \mathcal{L} and \mathcal{F} are of rank 1 and $n - 1$ respectively and $s \in \Gamma(X, \mathcal{E})$ is defined by $s \in \Gamma(X, \mathcal{L})$. Then, the right hand side equals $c_{n-1}(\mathcal{F}) \cap [D]$ by [4, Proposition (1.1) (iii)]. The left hand side also equals $c_{n-1}(\mathcal{F}) \cap [D]$ by the excess intersection formula [7, Theorem 6.3]. \square

We define the specialization of a cycle. Let $f: X \rightarrow Y$ be a smooth morphism of smooth schemes over a perfect field k and assume that X (resp. Y) is equidimensional of dimension $n+1$ (resp. 1). Let $y \in Y$ be a closed point, $V = Y - \{y\}$ be the complement and $U = X \times_Y V$ be the inverse image. Let $C \subset T^*U$ be a closed conical subset equidimensional of dimension $n + 1$ and assume that $U \rightarrow V$ is properly C -transversal. We define its specialization

$$\mathrm{sp}_y C \subset T^*X_y$$

as follows. By the assumption that $U \rightarrow V$ is properly C -transversal and [16, Lemma 3.1], the morphism $T^*U \rightarrow T^*U/V$ to the relative cotangent bundle is finite on C . Hence its image $C' \subset T^*U/V$ is a closed conical subset. Let $\bar{C}' \subset T^*X/Y$ be the closure and define $\mathrm{sp}_y C \subset T^*X_y$ to be the fiber $\bar{C}' \times_Y y \subset T^*X/Y \times_Y y = T^*X_y$. The specialization $\mathrm{sp}_y C \subset T^*X_y$ is a closed conical subset equidimensional of dimension n .

For a linear combination $A = \sum_a m_a C_a$ of irreducible components of $C = \bigcup_a C_a$, we define its specialization

$$\mathrm{sp}_y A \in Z_n(\mathrm{sp}_y C)$$

as follows. First, we define $A' \in Z_{n+1}(C')$ as the push-forward of A by the morphism $T^*U \rightarrow T^*U/V$ finite on C . Let $\bar{A}' \in Z_{n+1}(\bar{C}')$ be the unique element extending $A' \in Z_{n+1}(C')$. Then, we define $\mathrm{sp}_y A \in Z_n(\mathrm{sp}_y C)$ to be the *minus* of the pull-back of A' by the Gysin map for the immersion $i_y: X_y \rightarrow X$. If $X \rightarrow Y$ is proper, for a closed point $v \in V$ and the closed immersion $i_v: X_v \rightarrow X$, we have

$$(2.7) \quad (\mathrm{sp}_y A, T_{X_y}^* X_y)_{T^*X_y} = (i_v^! A, T_{X_v}^* X_v)_{T^*X_v}$$

since the definition of $i_v^! A$ in [16, Definition 7.1] involves the sign $(-1)^{\dim X - \dim X_v} = -1$.

Lemma 2.1.5. *Let $f: X \rightarrow Y$ be a smooth morphism of smooth schemes over a perfect field k and assume that X (resp. Y) is equidimensional of dimension $n + 1$ (resp. 1). Let $y \in Y$ be a closed point, $i_y: X_y \rightarrow X$ be the closed immersion of the fiber, $V = Y - \{y\}$ be the complement and $U = X \times_Y V$ be the inverse image. Let $C \subset T^*X$ be a closed conical subset equidimensional of dimension $n + 1$ such that $f: X \rightarrow Y$ is properly C -transversal.*

1. *For the restriction C_U of C on U , we have*

$$(2.8) \quad \mathrm{sp}_y C_U = i_y^\circ C.$$

2. *For a linear combination $A = \sum_a m_a C_a$ of irreducible components of $C = \bigcup_a C_a$ and its restriction A_U on U , we have*

$$(2.9) \quad \mathrm{sp}_y A_U = i_y^! A.$$

Proof. 1. By the assumption that $f: X \rightarrow Y$ is properly C -transversal and [16, Lemma 3.1], the morphism $T^*X \rightarrow T^*X/Y$ to the relative cotangent bundle is finite on C and hence its image $C' \subset T^*X/Y$ is a closed conical subset. Further C' with reduced scheme structure is flat over Y . Hence it equals the closure of the restriction C'_U and we obtain (2.8).

2. We consider the cartesian diagram

$$\begin{array}{ccc} T^*X & \longrightarrow & T^*X/Y \\ \uparrow & \square & \uparrow \\ X_y \times_X T^*X & \longrightarrow & T^*X_y. \end{array}$$

The right hand side is the minus of the image of A by the push-forward and the pull-back via upper right. The left hand side is the minus of the image of A by the pull-back and the push forward via lower left. Hence the assertion follows from the projection formula [7, Theorem 6.2 (a)]. \square

2.2 Characteristic cycle of the direct image

Let k be a field and let Λ be a finite field of characteristic ℓ invertible in k . Let X be a smooth scheme over k such that every irreducible component is of dimension n . Let \mathcal{F} be a constructible complex of Λ -modules on X and $C = SS\mathcal{F}$ be the singular support. Then, every irreducible component C_a of $C = \bigcup_a C_a$ has the same dimension as X [2, Theorem 1.3 (ii)] and the base $B = C \cap T_X^*X \subset T_X^*X = X$ defined as the intersection with the 0-sections equals the support of \mathcal{F} [2, Lemma 2.1 (i)]. Let $f: X \rightarrow Y$ be a morphism of smooth schemes over k , proper on the support of \mathcal{F} . Then, we have an inclusion

$$(2.10) \quad SSRf_*\mathcal{F} \subset f_*SS\mathcal{F}$$

by [2, Lemma 2.2 (ii)].

We restate a conjecture from [17, Conjecture 1].

Conjecture 2.2.1. *Let $f: X \rightarrow Y$ be a morphism of smooth schemes over a perfect field k . Assume that every irreducible component of X (resp. of Y) is of dimension n (resp. m). Let \mathcal{F} be a constructible complex of Λ -modules on X and $C = SS\mathcal{F}$ be the singular support. Assume that f is proper on the support of \mathcal{F} . Then, we have*

$$(2.11) \quad CCRf_*\mathcal{F} = f_!CC\mathcal{F}$$

in $\mathrm{CH}_m(f_*SS\mathcal{F})$.

If $\dim f_*SS\mathcal{F} \leq m$, the equality (2.11) is an equality as cycles in $\mathrm{CH}_m(f_*SS\mathcal{F}) = Z_m(f_*SS\mathcal{F})$ without rational equivalence.

A weaker version of Conjecture 2.2.1 is proved in the case k is finite and X and Y are projective in [19] using ε -factors.

If $Y = \mathrm{Spec} k$, the equality (2.11) means the index formula

$$(2.12) \quad \chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T_X^*X)_{T^*X}$$

where the right hand side denotes the intersection number. Further if X is projective, the equality (2.12) is proved in [16, Theorem 7.13].

Lemma 2.2.2. *Let $f: X \rightarrow Y$ be a morphism of smooth schemes over k and let \mathcal{F} be a constructible complex of Λ -modules. Assume that $f: X \rightarrow Y$ is proper on the support of \mathcal{F} . Assume that every irreducible component of X (resp. of Y) is of dimension n (resp. m).*

1. *Let*

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ & \searrow f & \downarrow f' \\ & & Y \end{array}$$

be a commutative diagram of morphisms of smooth schemes over k . Then, we have

$$(2.13) \quad f_! CCF = f'_!(g_! CCF)$$

in $\mathrm{CH}_m(f_* SS\mathcal{F})$.

2. Assume that one of the following conditions (1) and (2) is satisfied:

(1) $f: X \rightarrow Y$ is an immersion.

(2) $f: X \rightarrow Y$ is quasi-projective and $SS\mathcal{F}$ -transversal.

Then, we have $\dim(f_* SS\mathcal{F}) \leq m = \dim Y$ and

$$(2.11) \quad CCRf_* \mathcal{F} = f_! CCF$$

in $Z_m(f_* SS\mathcal{F})$.

Proof. 1. It follows from Lemma 2.1.1.

2. The case (1) is proved in [16, Lemma 5.13.2]. We show the case (2). Since $f_* SS\mathcal{F}$ is a subset of the 0-section $T_Y^* Y$, we have $\dim(f_* SS\mathcal{F}) \leq m = \dim Y$. We may assume that Y is connected and affine and hence X is quasi-projective. Let $X \rightarrow P$ be an immersion to a projective space and factorize $X \rightarrow Y$ as the composition of an immersion $X \rightarrow Y \times P$ and the projection $Y \times P \rightarrow Y$. Then, by 1 and the case (1), we may assume that $f: X \rightarrow Y$ is projective and smooth.

By the assumption that $f: X \rightarrow Y$ is $SS\mathcal{F}$ -transversal, it is locally acyclic relatively to \mathcal{F} by Lemma 1.3.4.1. Since $f: X \rightarrow Y$ is proper, the direct image $Rf_* \mathcal{F}$ is locally constant by [10, 5.2.4]. By Lemma 1.3.5.2, there exists a dense open subscheme $V \subset Y$ such that $f: X \rightarrow Y$ is properly $SS\mathcal{F}$ -transversal on V . By [16, Lemma 5.11.1] and Lemma 2.1.2, it suffices to show the equality

$$(2.14) \quad \mathrm{rank} Rf_* \mathcal{F} = (i_y^! CCF, T_{X_y}^* X_y)_{T^* X_y}$$

for a closed point $y \in V$. Since $\mathrm{rank} Rf_* \mathcal{F} = \chi(X_{\bar{y}}, i_y^* \mathcal{F})$, the equality (2.14) follows from the compatibility $CCi_y^* \mathcal{F} = i_y^! CCF$ with the pull-back [16, Theorem 7.6] and the index formula [16, Theorem 7.13]. \square

We consider the case where Y is a smooth curve and $\dim f_* SS\mathcal{F} \leq 1$. We recall the definition of the Artin conductor and the description of the characteristic cycle of a sheaf on a curve. Let Y be a smooth irreducible curve over a perfect field k and let \mathcal{G} be a constructible complex of Λ -modules on Y . Let $V \subset Y$ be a dense open subscheme such that the restriction \mathcal{G}_V is locally constant i. e. the cohomology sheaf $\mathcal{H}^q \mathcal{G}_V$ is locally constant for every integer q . For a closed point $y \in Y$, the Artin conductor $a_y \mathcal{G}$ is defined by

$$(2.15) \quad a_y \mathcal{G} = \mathrm{rank} \mathcal{G}_V - \mathrm{rank} \mathcal{G}_{\bar{y}} + \mathrm{Sw}_y \mathcal{G}.$$

Here \bar{y} denotes a geometric point above y and Sw_y denotes the alternating sum of the Swan conductor. The characteristic cycle is given by

$$(2.16) \quad CC\mathcal{G} = - \left(\mathrm{rank} \mathcal{G}_V \cdot [T_Y^* Y] + \sum_{y \in Y - V} a_y \mathcal{G} \cdot [T_y^* Y] \right)$$

by [16, Lemma 5.11.3]. Here T_y^*Y is the fiber of y .

Let $f: X \rightarrow Y$ be a morphism of smooth schemes over a perfect field k and $y \in Y$ be a closed point. Assume that $\dim Y = 1$. Let \mathcal{F} be a constructible complex of Λ -modules on X . Assume that $f: X \rightarrow Y$ is proper on the support of \mathcal{F} . Under the assumption $\dim f_*SS\mathcal{F} \leq 1$, the equality $CCRf_*\mathcal{F} = f_!CC\mathcal{F}$ (2.11) in $Z_1(f_*SS\mathcal{F})$ is equivalent to the equality

$$(2.17) \quad -a_y Rf_*\mathcal{F} = (CC\mathcal{F}, df)_{T^*X, X_y}.$$

for every closed point $y \in Y - V$ by (2.16), Lemma 2.2.2.2 (2) and Lemma 2.1.3.2, where the right hand side is defined as in (2.5).

Theorem 2.2.3. *Let $f: X \rightarrow Y$ be a quasi-projective morphism of smooth schemes over a perfect field k and $y \in Y$ be a closed point. Assume that $\dim Y = 1$. Let \mathcal{F} be a constructible complex of Λ -modules on X . Assume that $f: X \rightarrow Y$ is proper on the support of \mathcal{F} and is properly $SS\mathcal{F}$ -transversal on a dense open subscheme $V \subset Y$. Then, we have*

$$(2.17) \quad -a_y Rf_*\mathcal{F} = (CC\mathcal{F}, df)_{T^*X, X_y}.$$

Proof. We may assume that k is algebraically closed. By the same argument as in the proof of Lemma 2.2.2.2, we may assume that $f: X = Y \times P \rightarrow Y$ is the projection for a projective space P . By Lemma 2.1.3.1 and by replacing Y by a projective smooth curve over k containing Y as a dense open subscheme, we may assume that Y is projective and smooth.

By Lemma 2.2.2 applied to $X \rightarrow Y \rightarrow \text{Spec } k$, we obtain

$$(f_!CC\mathcal{F}, T_Y^*Y)_{T^*Y} = (CC\mathcal{F}, T_X^*X)_{T^*X}.$$

By the index formula [16, Theorem 7.13], we have

$$(CCRf_*\mathcal{F}, T_Y^*Y)_{T^*Y} = \chi(Y_{\bar{k}}, Rf_*\mathcal{F}) = \chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T_X^*X)_{T^*X}.$$

Thus, we have

$$(CCRf_*\mathcal{F} - f_!CC\mathcal{F}, T_Y^*Y)_{T^*Y} = 0.$$

Since the coefficients of T_Y^*Y in $CCRf_*\mathcal{F}$ and $f_!CC\mathcal{F}$ are equal by (2.4), (2.16) and the index formula [16, Theorem 7.13], we obtain

$$(2.18) \quad \sum_{y \in Y - V} -a_y Rf_*\mathcal{F} = \sum_{y \in Y - V} (CC\mathcal{F}, df)_{T^*X, X_y}.$$

By dévissage using Lemma 1.3.4.3 and [16, Lemma 5.13.1], we may assume that \mathcal{F} is a perverse sheaf. Set $\tilde{V} = V \cup \{y\}$ and $Z = Y - \tilde{V}$. By Corollary 1.1.3, Corollary 1.5.4 and weak approximation, there exists a faithfully flat finite morphism $Y' \rightarrow Y$ of projective smooth curves *étale at y* satisfying the following condition: Let

$$\begin{array}{ccccc} X & \longleftarrow & X' & \xleftarrow{\tilde{j}'} & X'_{\tilde{V}'} \\ f \downarrow & \square & f' \downarrow & \square & \downarrow \\ Y & \longleftarrow & Y' & \longleftarrow & \tilde{V}' = Y' \times_Y \tilde{V} \end{array}$$

be a cartesian diagram and set $\mathcal{F}' = j_{!*}' \mathcal{F}'_{\tilde{Y}'}$ for the pull-back $\mathcal{F}'_{\tilde{Y}'}$ of \mathcal{F} on $X'_{\tilde{Y}'}$. Then on $Y'_0 = Y' \times_Y Y$, the morphism $f': X' \rightarrow Y'$ is $SS\mathcal{F}'$ -transversal and hence is universally locally acyclic relatively to \mathcal{F}' .

For each $y' \in Z' = Z \times_Y Y'$, we have $a_{y'} Rf'_* \mathcal{F}' = (CC\mathcal{F}', df')_{T^*X', y'} = 0$. Since $Y' \rightarrow Y$ is étale at y , for each $y' \in Y' \times_Y y$, we have $a_y Rf_* \mathcal{F} = a_{y'} Rf'_* \mathcal{F}'$ and $(CC\mathcal{F}, df)_{T^*X, X_y} = (CC\mathcal{F}', df')_{T^*X', y'}$. Thus, by applying (2.18) to $f': X' \rightarrow Y'$ and \mathcal{F}' , we obtain

$$-[Y' : Y] \cdot a_y Rf_* \mathcal{F} = [Y' : Y] \cdot (CC\mathcal{F}, df)_{T^*X, y}$$

and hence (2.17). \square

Corollary 2.2.4 (cf. [4, Conjecture]). *Let $f: X \rightarrow Y$ be a projective flat morphism of smooth schemes over a perfect field k . Assume that $\dim X = n$, $\dim Y = 1$ and that there exists a dense open subscheme $V \subset Y$ such that the base change $f_V: X \times_Y V \rightarrow V$ is smooth. Then, for a closed point $y \in Y$, we have*

$$(2.19) \quad -a_y Rf_* \Lambda = (-1)^n c_{n, X_y}^X (\Omega_{X/Y}^1 \cap [X]).$$

Proof. Applying Theorem 2.2.3 to the constant sheaf $\mathcal{F} = \Lambda$ and $CC\Lambda = (-1)^n [T_X^* X]$, we obtain $-a_y Rf_* \Lambda = (-1)^n (T_X^* X, df)_{T^*X, X_y}$. By applying Lemma 2.1.4 to the right hand side and $[f^* \Omega_{Y/k}^1 \rightarrow \Omega_{X/k}^1]$, we obtain (2.19). \square

Theorem 2.2.5. *Let $f: X \rightarrow Y$ be a morphism of smooth schemes over a perfect field k . Let \mathcal{F} be a constructible complex of Λ -modules on X and $C = SS\mathcal{F}$ be the singular support. Assume that Y is projective, that $f: X \rightarrow Y$ is quasi-projective and is proper on the support of \mathcal{F} and that we have an inequality*

$$(2.20) \quad \dim f_* C \leq \dim Y = m.$$

Then, we have

$$(2.21) \quad CCRf_* \mathcal{F} = f_! CC\mathcal{F}$$

in $Z_m(f_ SS\mathcal{F})$.*

Proof. We may assume that k is algebraically closed. Since X is quasi-projective, there exists a locally closed immersion $i: X \rightarrow P$ to a projective space P . By decomposing f as the composition of the immersion $(i, f): X \rightarrow P \times Y$ and the second projection $P \times Y \rightarrow Y$, we may assume that f is projective and smooth by Lemma 2.2.2. Set $C = f_* SS\mathcal{F} \subset T^*Y$. We have $SSRf_* \mathcal{F} \subset f_* SS\mathcal{F} = C$. By the assumption, we have $\dim C \leq m$. By the index formula [16, Theorem 7.13] and Theorem 2.2.3, the equality (2.21) is proved for Y of dimension ≤ 1 . We show the general case by reducing to the case $\dim Y = 1$.

We take a closed immersion of Y to a projective space $i: Y \rightarrow \mathbf{P}$. We use the notations $\mathbf{P} \xleftarrow{p} Q \xrightarrow{p^\vee} \mathbf{P}^\vee$ in (1.4) and let $p_X: X \times_{\mathbf{P}} Q \rightarrow X$ be the projection. After replacing the immersion i by the composition with a Veronese embedding if necessary, we may assume that the restriction to $\mathbf{P}(i_* C) \subset Q = \mathbf{P}(T^*\mathbf{P})$ of the projection $p^\vee: Q \rightarrow \mathbf{P}^\vee$ is generically radicial by the assumption $\dim C \leq m = \dim Y$ and by [16, Corollary 3.21]. Let $C^\vee = p^\vee p_Y^* C \subset T^*\mathbf{P}^\vee$ and let D denote the image $p^\vee(\mathbf{P}(i_* C)) \subset \mathbf{P}^\vee$. By Lemma 1.2.7, Lemma 1.2.3.3 and the Bertini theorem, there exists a line $L \subset \mathbf{P}^\vee$ satisfying the following conditions: The immersion $h: L \rightarrow \mathbf{P}^\vee$ is properly C^\vee -transversal. The

morphism $h: L \rightarrow \mathbf{P}^\vee$ meets $p_X^\circ SS\mathcal{F}$ properly. The axis A_L of L meets Y transversely and L meets D transversely.

Since A_L meets Y transversely, the blow-up Y' of Y at $Y \cap A_L$ is smooth. We consider the cartesian diagram

$$(2.22) \quad \begin{array}{ccccc} X & \xleftarrow{p_X} & X \times_{\mathbf{P}} Q & \xleftarrow{h_X} & X' \\ f \downarrow & \square & \tilde{f} \downarrow & \square & \downarrow f' \\ Y & \xleftarrow{p_Y} & Y \times_{\mathbf{P}} Q & \xleftarrow{h_Y} & Y' \\ & & p^\vee \downarrow & \square & \downarrow p_L \\ & & \mathbf{P}^\vee & \xleftarrow{h} & L \end{array}$$

of projective smooth schemes over k . The equality (2.21) is equivalent to

$$p_!^\vee p_Y^! CCRf_* \mathcal{F} = p_!^\vee p_Y^! f_! CC\mathcal{F}.$$

It suffices to compare the coefficients of $C_a^\vee = p_Y^\vee p_Y^\circ C_a$ for each irreducible component of $C = \bigcup_a C_a$ of dimension $m = \dim Y$. Hence, this is further equivalent to

$$(2.23) \quad h^! p_!^\vee p_Y^! CCRf_* \mathcal{F} = h^! p_!^\vee p_Y^! f_! CC\mathcal{F}$$

since $\mathbf{P}(i_\circ C) \rightarrow D$ is generically radicial, $h: L \rightarrow \mathbf{P}^\vee$ is properly C^\vee -transversal and L meets D transversely. Let $\pi_X: X' \rightarrow X$ denote the composition $p_X \circ h_X$ of the top line in (2.22). We show that the equality (2.23) is equivalent to

$$(2.24) \quad CCR(p_L f')_* \pi_X^* \mathcal{F} = (p_L f')_! CC\pi_X^* \mathcal{F}.$$

First, we compare the left hand sides. By [16, Corollary 7.12] applied to $i_* Rf_* \mathcal{F}$ on \mathbf{P} , the left hand side of (2.23) equals $h^! CCRp_Y^\vee p_Y^* Rf_* \mathcal{F}$. Since $SSRp_Y^\vee p_Y^* Rf_* \mathcal{F} \subset C^\vee$ and since $h: L \rightarrow \mathbf{P}^\vee$ is properly C^\vee -transversal, the left hand side further equals $CC h^* R p_Y^\vee p_Y^* Rf_* \mathcal{F}$ by [16, Theorem 7.6]. By proper base change theorem, this is equal to the left hand side $CCR(p_L f')_* \pi_X^* \mathcal{F}$ of (2.24).

Next, we compare the right hand sides. The right hand side of (2.23) is equal to $(p_L f')_! \pi_X^! CC\mathcal{F}$ by the projection formula [7, Theorem 6.2 (a)]. Since $h: L \rightarrow \mathbf{P}^\vee$ is C^\vee -transversal and $C^\vee = p_Y^\vee p_Y^\circ C = (p^\vee \tilde{f})_* p_X^\circ SS\mathcal{F}$, the immersion $h_X: X' \rightarrow X \times_{\mathbf{P}} Q$ is $p_X^\circ SS\mathcal{F}$ -transversal by Lemma 1.2.6.2. Further since $h: L \rightarrow \mathbf{P}^\vee$ meets $p_X^\circ SS\mathcal{F}$ properly, the immersion $h_X: X' \rightarrow X \times_{\mathbf{P}} Q$ is properly $p_X^\circ SS\mathcal{F}$ -transversal. Since p_X is smooth, the composition $\pi_X = p_X \circ h_X$ is properly $SS\mathcal{F}$ -transversal. Thus by [16, Theorem 7.6], it further equals to the right hand side $(p_L f')_! CC\pi_X^* \mathcal{F}$ of (2.24).

We show the equality (2.24) by applying Theorem 2.2.3 to complete the proof. Since $\mathbf{P}(i_\circ C) \subset Y \times_{\mathbf{P}} Q = \mathbf{P}(Y \times_{\mathbf{P}} T^*\mathbf{P})$ is the complement of the largest open subset where $p_Y^\vee: Y \times_{\mathbf{P}} Q \rightarrow \mathbf{P}^\vee$ is $p_Y^\circ C$ -transversal and since $p_Y^\vee C = p_Y^\circ f_* SS\mathcal{F} = \tilde{f}_* p_X^\circ SS\mathcal{F} = \tilde{f}_* SS p_X^* \mathcal{F}$, the composition $p^\vee \tilde{f}: X \times_{\mathbf{P}} Q \rightarrow \mathbf{P}^\vee$ is $SS p_X^* \mathcal{F}$ -transversal on the complement $\mathbf{P}^\vee - D$ by [16, Lemma 3.8 (1) \Rightarrow (2)]. By Lemma 1.2.6, the morphism $p_L f': X' \rightarrow L$ is $SS\pi_X^* \mathcal{F}$ -transversal on the dense open subset $L - L \cap D$. Hence the equality (2.24) follows from Theorem 2.2.3 applied to $\pi_X^* \mathcal{F}$ and the equality (2.21) is proved. \square

In the case of characteristic 0, we recover the classical result as in [12, Proposition 9.4.2], in a slightly weaker form. Let X be a smooth scheme equidimensional of dimension

n over a field k and let $\omega_X \in \Omega^2(T^*X)$ denote the canonical symplectic form on the cotangent bundle T^*X . Let $C \subset T^*X$ be a closed conical subset. We say that C is *isotropic* if the restriction of ω_X on C is 0. We say that C is *Lagrangian* if it is isotropic and if C is equidimensional of dimension n .

Lemma 2.2.6. *Let k be a field of characteristic 0 and let $f: X \rightarrow Y$ be a morphism of smooth schemes over k . Assume that X (resp. Y) is equidimensional of dimension n (resp. m). Let $C \subset T^*X$ be a closed conical subset. If $C \subset T^*X$ is isotropic, then $f_*C \subset T^*Y$ is also isotropic.*

The author learned the following proof from Beilinson.

Proof. Let $T_\Gamma^*(X \times Y) \subset T^*(X \times Y)$ be the normal bundle of the graph $\Gamma \subset X \times Y$ of $f: X \rightarrow Y$ and let $p_2: T^*(X \times Y) = T^*X \times T^*Y \rightarrow T^*Y$ be the projection. The direct image $f_*C \subset T^*Y$ equals the image by p_2 of the intersection $C_1 = T_\Gamma^*(X \times Y) \cap (C \times T^*Y)$.

Since the normal bundle $T_\Gamma^*(X \times Y) \subset T^*(X \times Y)$ is isotropic and since $\omega_{X \times Y}$ equals the sum $p_1^*\omega_X + p_2^*\omega_Y$ of the pull-backs by projections, the assumption that $C \subset T^*X$ is isotropic implies that the restriction of $p_2^*\omega_Y$ on C_1 is 0. Since k is of characteristic 0, for each irreducible component C' of $f_*C \subset T^*Y$, there exists a closed subset $C'_1 \subset C_1$ generically étale over C' . Hence the assertion follows. \square

Proposition 2.2.7. *Let k be a field of characteristic 0 and let X be a smooth schemes over k . Let \mathcal{F} be a constructible complex of Λ -modules on X .*

1. *The singular support $SS\mathcal{F}$ is Lagrangian.*
2. *Let $f: X \rightarrow Y$ be a morphism of smooth schemes over k . Assume that f is proper on the support of \mathcal{F} . Then, the inequality (2.20) holds. Further if $f: X \rightarrow Y$ is quasi-projective, the equality (2.21) holds.*

Proof. 1. We may assume that X is equidimensional of dimension n . Since the singular support $SS\mathcal{F}$ is equidimensional of dimension n [2, Theorem 1.3 (ii)], it suffices to show that $SS\mathcal{F}$ is isotropic. By devissage, we may assume that there exist a locally closed immersion $i: V \rightarrow X$ of smooth scheme, a locally constant sheaf \mathcal{G} on V and $\mathcal{F} = i_!\mathcal{G}$.

Since the resolution of singularity is known in characteristic 0, the immersion i is decomposed by an open immersion $j: V \rightarrow W$ and a proper morphism $h: W \rightarrow X$ such that W is smooth and V is the complement of a divisor with simple normal crossings. Thus, by the inclusion $SS\mathcal{F} = SSRh_*j_!\mathcal{G} \subset h_*SSj_!\mathcal{G}$ and Lemma 2.2.6, it is reduced to the case where $i = j$ is an open immersion of the complement of a divisor with simple normal crossings. Since k is of characteristic 0, this case is proved in [16, Proposition 4.11].

2. By 1 and Lemma 2.2.6, the direct image $f_*SS\mathcal{F}$ is isotropic. Hence the inequality $\dim f_*SS\mathcal{F} \leq \dim Y$ (2.20) holds.

We show the equality $CCRf_*\mathcal{F} = f_!CC\mathcal{F}$ (2.21). Similarly as in the proof of Theorem 2.2.5, we may assume that Y is affine and $f: X = P \times Y \rightarrow Y$ is the projection for a projective smooth scheme P over k . By resolution of singularity, we may assume that Y is projective and smooth. Then since the inequality (2.20) holds, we may apply Theorem 2.2.5. \square

2.3 Index formula for vanishing cycles

We prepare some notation to formulate an index formula for vanishing cycle complex. Let $f: X \rightarrow Y$ be a smooth morphism of smooth schemes over a perfect field k . Assume that

X (resp. Y) is equidimensional of dimension $n + 1$ (resp. 1). Let \mathcal{F} be a constructible complex of Λ -modules on X . Let $y \in Y$ be a closed point and $i_y: X_y \rightarrow X$ be the closed immersion of the fiber. Assume that $f: X \rightarrow Y$ is properly $SS\mathcal{F}$ -transversal on the complement $X - X_y$ of the fiber $X_y = f^{-1}(y)$. Then, the specialization

$$(2.25) \quad \mathrm{sp}_y SS\mathcal{F} \subset T^*X_y$$

is defined as a closed conical subset equidimensional of dimension n . Further, the specialization

$$(2.26) \quad \mathrm{sp}_y CC\mathcal{F} \in Z_n(\mathrm{sp}_y SS\mathcal{F})$$

is defined as a cycle.

Lemma 2.3.1. *Let $f: X \rightarrow Y$ be a smooth morphism of smooth schemes over a field k and assume $\dim Y = 1$. Let \mathcal{F} be a constructible complex of Λ -modules on X and assume that $f: X \rightarrow Y$ is properly $SS\mathcal{F}$ -transversal. Let $y \in Y$ be a closed point. Then, we have*

$$(2.27) \quad SSR\Psi_y \mathcal{F} = \mathrm{sp}_y SS\mathcal{F}.$$

Further if k is perfect, we have

$$(2.28) \quad CCR\Psi_y \mathcal{F} = \mathrm{sp}_y CC\mathcal{F}.$$

Proof. Let $i_y: X_y \rightarrow X$ denote the closed immersion of the fiber. Then, by the assumption that $f: X \rightarrow Y$ is properly $SS\mathcal{F}$ -transversal, we have $\mathrm{sp}_y SS\mathcal{F} = i_y^\circ SS\mathcal{F}$ and $\mathrm{sp}_y CC\mathcal{F} = i_y^! CC\mathcal{F}$. Recall that the definitions of sp_y and $i_y^!$ both involve the minus sign.

Since $f: X \rightarrow Y$ is locally acyclic relatively to \mathcal{F} by Lemma 1.3.4.2, the canonical morphism $i_y^* \mathcal{F} \rightarrow R\Psi_y \mathcal{F}$ is an isomorphism. Hence the equalities (2.27) and (2.28) follow from Lemma 1.3.1 and [16, Theorem 7.6] respectively. \square

The following example shows that the inclusion $SSR\Psi \mathcal{F} \subset \mathrm{sp}_y SS\mathcal{F}$ does not hold in general.

Example 2.3.2. Let k be a field of characteristic $p > 2$. Let $X = \mathbf{A}^1 \times \mathbf{P}^1$ and $j: U = \mathbf{A}^1 \times \mathbf{A}^1 = \mathrm{Spec} k[x, y] \rightarrow X$ be the open immersion. Let \mathcal{G} be the locally constant sheaf of Λ -modules of rank 1 on U defined by the Artin-Schreier covering $t^p - t = x^p y^2$ and by a non-trivial character $\mathbf{F}_p \rightarrow \Lambda^\times$. Then, the nearby cycles complex $R\Psi_\infty \mathcal{F}$ is acyclic except at the closed point $(0, \infty)$ or at degree 1 and $\dim R^1 \Psi_{\mathcal{F}(0, \infty)} = 1$. Hence, the singular support $SSR\Psi_\infty \mathcal{F}$ equals the fiber $T_{(0, \infty)}^* X_\infty$ at the closed point and is not a subset of the zero-section $\mathrm{sp}_\infty SS\mathcal{F} = T_{X_\infty}^* X_\infty$.

Let $Z \subset X_y$ be a closed subset. Assume that $f: X \rightarrow Y$ is properly $SS\mathcal{F}$ -transversal on the complement of Z . Then, on the complement $X_y - Z$, we have $\mathrm{sp}_y CC\mathcal{F} = i_y^! CC\mathcal{F} = CCi_y^* \mathcal{F}$ by Lemma 2.1.5 and the compatibility with the pull-back [16, Theorem 7.6]. Thus, the difference

$$(2.29) \quad \delta_y CC\mathcal{F} = \mathrm{sp}_y CC\mathcal{F} - CCi_y^* \mathcal{F}$$

is defined as a cycle in $Z_n(Z \times_X (\mathrm{sp}_y SS\mathcal{F} \cup SSR\Psi_y^* \mathcal{F}))$ supported on Z . If Z is proper over Y , the intersection number $(\delta_y SS\mathcal{F}, T_{X_y}^* X_y)_{T^* X_y}$ is defined.

Proposition 2.3.3. *Let $f: X \rightarrow Y$ be a smooth morphism of smooth schemes over a perfect field k . Assume that X (resp. Y) is equidimensional of dimension $n+1$ (resp. 1). Let \mathcal{F} be a constructible complex of Λ -modules on X . Let $y \in Y$ be a closed point and let $Z \subset X_y$ be a closed subset. Assume that $f: X \rightarrow Y$ is properly $SS\mathcal{F}$ -transversal on the complement of Z and that either of the following conditions (1) and (2) is satisfied:*

(1) $f: X \rightarrow Y$ is projective.

(2) $\dim Z = 0$.

Then, for the vanishing cycles complex $R\Phi_y\mathcal{F}$, we have

$$(2.30) \quad \chi(Z_{\bar{k}}, R\Phi_y\mathcal{F}) = (\delta_y CC\mathcal{F}, T_{X_y}^* X_y)_{T^* X_y}.$$

Proof. We may assume that k is algebraically closed.

We show the case (1). Let $v \in Y$ be a closed point different from y and let $i_v: X_v \rightarrow X$ be the closed immersion. Then, since the projective morphism $f: X \rightarrow Y$ is locally acyclic relative to \mathcal{F} outside Z by Lemma 1.3.4.2, the left hand side of (2.30) equals

$$(2.31) \quad \chi(Z, R\Phi_y\mathcal{F}) = \chi(X_y, R\Psi_y\mathcal{F}) - \chi(X_y, i_y^*\mathcal{F}) = \chi(X_v, i_v^*\mathcal{F}) - \chi(X_y, i_y^*\mathcal{F})$$

The right hand side of (2.30)

$$(\delta_y CC\mathcal{F}, T_{X_y}^* X_y)_{T^* X_y} = (\mathrm{sp}_y CC\mathcal{F}, T_{X_y}^* X_y)_{T^* X_y} - (CCi_y^*\mathcal{F}, T_{X_y}^* X_y)_{T^* X_y}$$

equals

$$(2.32) \quad (i_v^! CC\mathcal{F}, T_{X_v}^* X_v)_{T^* X_v} - (CCi_y^*\mathcal{F}, T_{X_y}^* X_y)_{T^* X_y}$$

by (2.7). Since $i_v: X_v \rightarrow X$ is properly $SS\mathcal{F}$ -transversal by Lemma 1.2.6, the right hand side of (2.31) equals (2.32) by the compatibility with the pull-back [16, Theorem 7.6] and the index formula [16, Theorem 7.13]. Thus the equality (2.30) is proved.

We show the case (2). Since the formation of nearby cycles complex commutes with base change by [6, Proposition 3.7], we may assume that the action of the inertia group I_y on $R\Psi_y\mathcal{F}$ is trivial. Since the vanishing cycles functor is t -exact by [11, Corollaire 4.6], we may assume that \mathcal{F} is a simple perverse sheaf.

First, we consider the case \mathcal{F} is supported on the closed fiber X_y . By the assumption that $f: X \rightarrow Y$ is properly $SS\mathcal{F}$ -transversal on the complement of Z , the morphism $f: X \rightarrow Y$ is locally acyclic relative to \mathcal{F} on the complement of Z . Thus \mathcal{F} is supported on Z and the assertion follows in this case.

We may assume that the restriction $\mathcal{F}|_{X_\eta}$ on the generic fiber is non-trivial. Then, by Proposition 1.1.2.2, the morphism $f: X \rightarrow Y$ is locally acyclic relative to \mathcal{F} . Hence by Lemma 1.3.8.2, the morphism $f: X \rightarrow Y$ is properly $SS\mathcal{F}$ -transversal and the assertion follows from Lemma 2.3.1. \square

In the case (2) $\dim Z = 0$, Proposition 2.3.3 means $CCR\Phi_y\mathcal{F} = \delta_y CC\mathcal{F}$. However, Examples 1.4.8 and 2.3.2 show that one cannot expect to have $CCR\Psi_y\mathcal{F} = \mathrm{sp}_y CC\mathcal{F}$ or equivalently $CCR\Phi_y\mathcal{F} = \delta_y CC\mathcal{F}$ in general.

References

- [1] M. Artin, *Théorème de finitude pour un morphisme propre; dimension cohomologique des schémas algébriques affines*, SGA 4 Exposé XIV, Théorie des Topos et Cohomologie Étale des Schémas, Lecture Notes in Mathematics Volume 305, 1973, pp 145-167.

- [2] A. Beilinson, *Constructible sheaves are holonomic*, Selecta Math. New Ser. 22, Issue 4, (2016) 1797-1819.
- [3] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Analyse et topologie sur les espaces singuliers (I), Astérisque 100, (1982).
- [4] S. Bloch, *Cycles on arithmetic schemes and Euler characteristics of curves*, Algebraic geometry, Bowdoin, 1985, 421-450, Proc. Symp. Pure Math. 46, Part 2, Am. Math. Soc., Providence, RI (1987).
- [5] A. J. de Jong, *Smoothness, semi-stability and alterations*, Publ. Math. IHÉS, 83 (1996), 51-93.
- [6] P. Deligne, *Théorèmes de finitude en cohomologie ℓ -adique*, Cohomologie étale, SGA 4 $\frac{1}{2}$, Springer Lecture Notes in Math. 569, (1977), 233–251.
- [7] W. Fulton, *INTERSECTION THEORY*, 2nd edition (1998) Springer.
- [8] A. Grothendieck, *ÉLÉMENTS DE GÉOMÉTRIE ALGÈBRE IV, Étude locale des schémas et des morphismes de schémas*, Publ. Math. IHES 20, 24, 28, 32 (1964-67).
- [9] H. Hu, E. Yang, *Relative singular support and the semi-continuity of characteristic cycles for étale sheaves*, [arXiv:1702.06752](https://arxiv.org/abs/1702.06752)
- [10] L. Illusie, *Appendice à Théorèmes de finitude en cohomologie ℓ -adique*, Cohomologie étale SGA 4 $\frac{1}{2}$, Springer Lecture Notes in Math. 569 (1977) 252–261.
- [11] —, *Autour du théorème de monodromie locale*, Astérisque 223 (1994), 9-57, Périodes p -adiques (Bures-sur-Yvette, 1988).
- [12] M. Kashiwara, P. Schapira, *SHEAVES ON MANIFOLDS*, Springer-Verlag, Grundlehren der Math. Wissenschaften 292, (1990).
- [13] K. Kato, T. Saito, *On the conductor formula of Bloch*, Publ. Math., IHES 100 (2004), 5-151.
- [14] N. Katz, G. Laumon, *Transformation de Fourier et majoration de sommes exponentielles*, Publ. Math. IHÉS (1985) 62, pp 145-202.
- [15] F. Orgogozo, *Modifications et cycles évanescents sur une base de dimension supérieure à un*, Int. Math. Res. Notices, (2006) No. 13, Article ID 25315, 1-38.
- [16] T. Saito, *The characteristic cycle and the singular support of a constructible sheaf*, Inventiones Math. 207(2) (2017), 597-695.
- [17] —, *On the proper push-forward of the characteristic cycle of a constructible sheaf*, preprint, <https://arxiv.org/abs/1607.03156>.
- [18] M. Temkin, *Stable modification of relative curves*, J. of Algebraic Geometry, 19 (2010) 603-677.
- [19] N. Umezaki, E. Yang, Y. Zhao, *Characteristic class and the ε -factor of an étale sheaf*, [arXiv:1701.02841](https://arxiv.org/abs/1701.02841)